MSC 65L70, 65L07

Connective estimates of errors in the linearization of nonlinear systems

O.V. Druzhinina [1], A.V. Shchennikov [2], E.V. Shchennikova [2], V.N. Shchennikov [2]

Federal Research Center Computer Science and Control of the Russian Academy of Sciences [1], National Research Ogarev Mordovia State University [2]

Abstract: The article describes differential equations with structural disturbance of a special form. These systems can be connective asymptotic stable. For such systems we found estimates of linearization, which can be connective. The present work are articles [2] — [5].

Keywords: connective asymptotic stability, estimate of linearization error, structural disturbance.

1. Problem description

In this section we consider the system

$$\frac{dx_s}{dt} = f_s(t, x_s) + f_{1s}(t, x_1, \dots, x_q) + f_{2s}(t, x_1, \dots, x_q) + r_s(t) \tag{1}$$

where $f_s(t, x_s) \in C(J^+ \times R^{n_s} \to R^{n_s})$, $f_{is}(t, x_1, \dots, x_q) \in C(J^+ \times R^{n_1} \times \dots \times R^{n_q} \to R^{n_s})$, $r_s(t) \in C(J^+ \to R^{n_s})$, $J^+ = \{t : t \geqslant t_0 \geqslant 0\}$, $\|r_s(t)\| \leqslant k_s$, $0 < k_s = \text{const}$; $x = (x_1^T, \dots, x_q^T)^T$, $R^{n_1} \oplus \dots \oplus R^{n_q} = R^n$, $s = \overline{1, q}$. Superscript ^{T} denotes transposition. We will take Euclidean norm as the norm of a vector.

Assume $(x_1^T(t_0), \dots, x_q^T(t_0))^T = x(t_0) = x_0$. Suppose that with $r_s(t) \equiv 0$ $(s = \overline{1, q})$ the system (1) has a unique equilibrium $x_1 = \dots = x_q = 0$, and besides $f_s(t, 0) \equiv f_{1s}(t, 0, \dots 0) \equiv f_{2s}(t, 0, \dots 0) = 0$, $s = \overline{1, q}$. We also assume that the vector functions $f_s(t, x_s)$, $f_{is}(t, x_1, \dots, x_q)$ $(i = 1, 2;\ s = \overline{1, q})$ are sufficiently smooth concerning phase variables. This ensures the existence and uniqueness of a solution of the Cauchy problem with initial data $(t_0, x_0) \in J^+ \times R^n$.

We assume that the "linear"approximation of the system (1) is a system

$$\frac{dp_s}{dt} = f_s(t, p_s) + f_{1s}(t, p_1, \dots, p_q) + r_s(t),\ s = \overline{1, q} \tag{2}$$

with initial data $(p_1^T(t_0), \dots, p_q^T(t_0))^T = p(t_0) = x_0 = x(t_0) = (x_1^T(t_0), \dots, x_q^T(t_0))^T$.

To obtain connective estimates of errors of linearization let us introduce fundamental coupling matrices $\overline{L} = (\overline{l}_{sj})_{1,1}^{q,q}$ and $\overline{E} = (\overline{e}_{sj})_{1,1}^{q,q}$ and current coupling matrices $L = (l_{sj})_{1,1}^{q,q}$, $E = (e_{sj})_{1,1}^{q,q}$, $L \in \overline{L}$, $E \in \overline{E}$. Here $\overline{e}_{sj} = \{0$ if there is no connection between the subsystems, and 1 if the connection between the subsystems takes place$\}$. Similar arguments are taking elements of matrix $\overline{E}$. Elements $l_{sj} = \{0$ if $\overline{l}_{sj} = 0$; 0 if $\overline{l}_{sj} = 1$, but the corresponding bond in the case at present; 1 otherwise$\}$. Similar arguments are taking elements of matrix E. More details of the systems, taking into account the fundamental matrices and current connections are described in [4, chapter 2]. Then the systems (1) and (2) will take following forms

$$\frac{dx_s}{dt} = f_s(t, x_s) + f_{1s}(t, l_{s1}x_1, \dots, l_{sq}x_q) + f_{2s}(t, e_{s1}x_1, \dots, e_{sq}x_q) + r_s(t),\ s = \overline{1, q} \tag{1_1}$$

and

$$\frac{dp_s}{dt} = f_s(t, p_s) + f_{1s}(t, l_{s1}p_1, \dots, l_{sq}p_q) + r_s(t),\ s = \overline{1, q} \tag{2_1}$$

.

Here $\overline{L}$ and $\overline{E}$ are fundamental coupling matrices of subsystems and the original system respectively; L and E, $L \in \overline{L}$, $E \in \overline{E}$ are current coupling matrices. It is obvious that under conditions imposed on the right-hand sides of systems (1) and (2) we may apply Caratheodory;s theorem on the existence and uniqueness of the Cauchy problem for systems (1_1) and (2_1) made in [5, chapter 1]. Hence solutions of the systems (1_1) and (2_1) are absolutely continuous vector functions.

2. Estimate constructing

We assume that the systems (1_1) and (2_1) admit representations

$$\frac{d\overline{x}_s}{dt} = f_{s1}(t, \overline{x}_s, \overline{\overline{x}}_s) + f_{1s}(t, \bar{l}_{s1}\overline{x}_1, \ldots, \bar{l}_{sq}\overline{x}_q, \bar{\bar{l}}_{s1}\overline{\overline{x}}_1, \ldots, \bar{\bar{l}}_{sq}\overline{\overline{x}}_q) + \tag{1_2}$$

$$+ f_{2s1}(t, \overline{e}_{s1}\overline{x}_1, \ldots, \overline{e}_{sq}\overline{x}_q, \overline{\overline{e}}_{s1}\overline{\overline{x}}_1, \ldots, \overline{\overline{e}}_{sq}\overline{\overline{x}}_q) + r_{s1}(t_0)$$

$$\frac{d\overline{\overline{x}}_s}{dt} = f_{s1}(t, \overline{x}_s, \overline{\overline{x}}_s) + f_{1s}(t, \bar{l}_{s1}\overline{x}_1, \ldots, \bar{l}_{sq}\overline{x}_q, \bar{\bar{l}}_{s1}\overline{\overline{x}}_1, \ldots, \bar{\bar{l}}_{sq}\overline{\overline{x}}_q) +$$

$$+ f_{2s1}(t, \overline{e}_{s1}\overline{x}_1, \ldots, \overline{e}_{sq}\overline{x}_q, \overline{\overline{e}}_{s1}\overline{\overline{x}}_1, \ldots, \overline{\overline{e}}_{sq}\overline{\overline{x}}_q) + r_{s2}(t), s = \overline{1, q}$$

and

$$\frac{d\overline{p}_s}{dt} = f_{s1}(t, \overline{p}_s, \overline{\overline{p}}_s) + f_{1s1}(t, \bar{l}_{s1}\overline{p}_1, \ldots, \bar{l}_{sq}\overline{p}_q, \bar{\bar{l}}_{s1}\overline{\overline{p}}_1, \ldots, \bar{\bar{l}}_{sq}\overline{\overline{p}}_q) + r_{s1}(t) \tag{2_2}$$

$$\frac{d\overline{\overline{p}}_s}{dt} = f_{s2}(t, \overline{p}_s, \overline{\overline{p}}_s) + f_{1s2}(t, \bar{l}_{s1}\overline{p}_1, \ldots, \bar{l}_{sq}\overline{p}_q, \bar{\bar{l}}_{s1}\overline{\overline{p}}_1, \ldots, \bar{\bar{l}}_{sq}\overline{\overline{p}}_q) + r_{s2}(t),\ s = \overline{1, q}.$$

Here

$$\overline{x}_s \in \mathbb{R}^{n'_s},\ \overline{\overline{x}}_s \in \mathbb{R}^{n''_s},\ \mathbb{R}^{n'_s} \times \mathbb{R}^{n''_s} = \mathbb{R}^{n_s},\ x_s = (\overline{x}_s^T, \overline{\overline{x}}_s^T)^T,\ \overline{x} = (\overline{x}_1^T, \ldots, \overline{x}_q^T)^T,\ \overline{\overline{x}} = (\overline{\overline{x}}_1^T, \ldots, \overline{\overline{x}}_q^T)^T,$$

$$\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_q} = \mathbb{R}^n,\ x = (\overline{x}^T, \overline{\overline{x}}^T)^T;\ f_{s1} \in C(J^+ \times \mathbb{R}^{n'_s} \times \cdots \times \mathbb{R}^{n''_s} \to \mathbb{R}^{n'_s}),$$

$$f_{s2} \in C(J^+ \times \mathbb{R}^{n'_s} \times \cdots \times \mathbb{R}^{n''_s} \to \mathbb{R}^{n''_s}),\ f_{is1} \in C(J^+ \times \mathbb{R}^{n'_1} \times \mathbb{R}^{n''_1} \times \cdots \times \mathbb{R}^{n'_q} \times \mathbb{R}^{n''_q} \to \mathbb{R}^{n'_s}),$$

$$f_{is2} \in C(J^+ \times \mathbb{R}^{n'_1} \times \mathbb{R}^{n''_1} \times \cdots \times \mathbb{R}^{n'_q} \times \mathbb{R}^{n''_q} \to \mathbb{R}^{n''_s}),\ r_s \in C(J^+ \to \mathbb{R}^{n_s}),$$

$$J^+ = \{t :\ t \geqslant t_0 \geqslant 0\},\ \|r_{si}(t)\| \leqslant k_{si} = \text{const},\ i = 1, 2,\ s = \overline{1, q}.$$

We also assume that the phase variables $p_1 \ldots p_n$ of the system (2_1) appear to be similar to the phase variables $x_1, \ldots, x_q$ and have some indices. Further assume that

a) for each subsystems

$$\frac{d\overline{p}_s}{dt} = f_{s1}(t, \overline{p}_s\overline{\overline{p}}_s) \tag{3}$$

$$\frac{d\overline{\overline{p}}_s}{dt} = f_{s2}(t, \overline{p}_s\overline{\overline{p}}_s).$$

there are Lyapunov functions satisfying

$$a_{1s}\omega(\|\overline{p}_s)\| \leqslant V_s(t, \overline{p}_s, \overline{\overline{p}}_s) \leqslant a_{2s}\omega(\|\overline{p}_s\|)$$

$$|V_s(t_2, \overline{p}_{s2}, \overline{\overline{p}}_{s2}) - V_s(t_1, \overline{p}_{s1}, \overline{\overline{p}}_{s1})| \leqslant L_s(|t_2 - t_1| + \|\overline{p}_{s2} - \overline{p}_{s1}\| + \|\overline{\overline{p}}_{s2} - \overline{\overline{p}}_{s1}\|), \tag{4}$$

$$D^+V_s(t, \overline{p}_s, \overline{\overline{p}}_s)|_{(3)} \leqslant -c_s\omega(\|\overline{p}_s\|),$$

where $D^+V_s(t,\overline{p}_s,\overline{\overline{p}}_s)|_{(3)} = \lim_{h\to+0}\sup \frac{1}{h}\left\{V_s(t+h,\overline{p}_s+hf_{s1}(t,\overline{p}_s,\overline{\overline{p}}_s),\overline{\overline{p}}_s+hf_{s2}(t,\overline{p}_s,\overline{\overline{p}}_s)\right\}, \omega(\|\overline{p}_s\|)$ – Khan function [6, §8.2] , $\omega(0)=0$, $t\in J^+$, $\overline{p}_s\in\mathbb{R}^{n'_s}$, $\overline{\overline{p}}_s\in\mathbb{R}^{n''_s}$; a_{1s},a_{2s},c_s – positive real constants, $s=\overline{1,q}$.

b) there are positive real constants α_s $(s=\overline{1,q})$ and the number $0<\rho<1$ such that satisfy inequality

$$\lim_{h\to+0}\sup\frac{1}{h}\{V_s(t+h,\overline{x}_s+h(f_{s1}(t,\overline{x}_s,\overline{\overline{x}}_s)+f_{1s1}(t,\bar{l}_{s1}\overline{x}_1,\dots,\bar{l}_{sq}\overline{x}_q,\bar{\bar{l}}_{s1}\overline{\overline{x}}_1,\dots,\bar{\bar{l}}_{sq}\overline{\overline{x}}_q),$$

$$\overline{\overline{x}}_s+h(f_{s2}(t,\overline{x}_s,\overline{\overline{x}}_s)+f_{1s2}(t,\bar{l}_{s1}\overline{x}_1,\dots,\bar{l}_{sq}\overline{x}_q,\bar{\bar{l}}_{s1}\overline{\overline{x}}_1,\dots,\bar{\bar{l}}_{sq}\overline{\overline{x}}_q))-V(t,\overline{x}_s,\overline{\overline{x}})_s\}\leqslant$$

$$\leqslant(\rho-1)\sum_{s=1}^{q}\alpha_s(L)\omega(\|\overline{x}_s\|).$$

Then based on the condition a) and b) we have

$$D^+V_s(t,\overline{x}_s,\overline{\overline{x}}_s)|_{(4.6.1_2)}\leqslant(\rho-1)\sum_{s=1}^{q}\alpha_s(L)\omega(\|\overline{\overline{x}}_s\|)+\sum_{s,j=1}^{q}(L_sA_{sj}e_{sj}\omega^2(\|\overline{\overline{x}}_j\|)+L_s(K_{s1}+K_{s2}))$$

For the system 1_2 we choose Lyapunov function in following form

$$V(t,x)=\sum_{s=1}^{q}d_sV_s(t,\overline{x}_s,\overline{\overline{x}}_s), \tag{5}$$

where $d_s>0$ - real numbers.
From the definition of $V(t,x)$ it follows that

$$D^+V(t,x)|_{(4.6.1_2)}\leqslant\sum_{s=1}^{q}d_s\{(\rho-1)\alpha_s(L)\omega(\|\overline{\overline{x}}_s\|)+$$

$$+\sum_{j=1}^{q}(L_sA_{sj}\omega^2(\|\overline{\overline{x}}_j\|)+L_sK_s)\}.$$

Note that for the definition of $V(t,x)$ in form (5)), correctness of inequation follows

$$\sum_{s=1}^{q}d_sa_{1s}\omega(\|\overline{x}_s\|)\leqslant V(t,x)\leqslant\sum_{s=1}^{q}d_sa_{2s}\omega(\|\overline{x}_s\|),$$

and, consequently,

$$a_1\omega(\|\overline{x}_s\|)\leqslant V_s(t,\overline{x}_s,\overline{\overline{x}}_s)\leqslant a_2\omega(\|\overline{x}_s\|), \tag{6}$$

where a_1 and a_2 are real positive constants. With estimates (6) and representation of ω we have

$$D^+V(t,x)|_{(1_2)}\leqslant(\rho-1)\alpha(L)\frac{V}{a_2}+\beta(E)\frac{1}{a_1^2}V^2+K, \tag{7}$$

where $\alpha=\sum_{s,j=1}^{q}d_s\alpha_sl_{sj}$, $\beta=\sum_{s,j=1}^{q}d_jA_{sj}e_{sj}$, $K=\sum_{s=1}^{q}d_sL_s(K_{s1}+K_{s2})$, and differential equation of comparison for (7) is given by

$$D^+U(t,x)=(\rho-1)\frac{\alpha(L)}{a_2}U+\frac{\beta(E)}{a_1^2}U^2+K, \tag{8}$$

$$U(t_0,x_0)=U_0=V_0=V(t_0,\overline{x}_0)=\sum_{s=1}^{q}\overline{d}V_s(t_0,\overline{x}_s(t_0),\overline{\overline{x}}_s(t_0)).$$

According to comparison theorem, we will have $V(t) \leqslant U(t)$ for $t \geqslant t_0$.
Let us presume that in field $U \geqslant 0$ the equation

$$H(U) = \frac{\beta(E)}{a_1^2}U^2 + (\rho - 1)\frac{\alpha(L)}{a_2}U + K = 0 \qquad (8_1)$$

has solutions. It is clear that there are no more than two solutions. We denote them by $U_1(a_1, a_2, \alpha(L), \beta(E))$ and $U_2(a_1, a_2, \alpha(L), \beta(E))$. Further they will be denoted by $U_1(\cdot)$ and $U_2(\cdot)$. Then

$$H(U) = \begin{cases} \geqslant 0, \ for \ 0 \leqslant U \leqslant U_1(\cdot), \ U \geqslant U_2(\cdot), \\ \leqslant 0, \ for \ U_1(\cdot) \leqslant U \leqslant U_2(\cdot). \end{cases}$$

Thereby $U_1(\cdot)$ is connectivly asymptotically stable equilibrium of the equation (8) and $U_2(\cdot)$ is unstable one. So, the solution $U(t, t_0, U_0)$ of the equation (8) approaches $U_1(\cdot)$ monotonically when $U_0 < U_2(\cdot)$ and $t - t_0 \to \infty$. Hence in $U_0 < U_2(\cdot)$ the solution of the equation (8) satisfies the valuation

$$\sup_{t \geqslant t_0} \|U(t, t_0, U_0)\| \max\{a_2\omega(\overline{x}_0), U_1(\cdot)\},$$

from which it follows upper estimate

$$\sup_{t \geqslant t_0} \|\overline{x}_s(t, t_0, U_0)\| \leqslant \omega^{-1}\left(\frac{1}{a_1}\max\{a_2\omega(\|\overline{x}_0\|), U_1(a_1, a_2, \alpha(L), \beta(E))\}\right). \qquad (9)$$

We shall now construct a connective errors estimate of linearization concerning part of phase variables $\overline{x}$. For that we introduce an accessory differential equation system based on a difference of the system solutions (1_2) and (2_2). Consider two vector functions $\overline{\varepsilon}(t) = \overline{x}(t, t_0, \overline{x}_0, \overline{\overline{x}}_0) - \overline{p}(t, t_0, \overline{p}_0, \overline{\overline{p}}_0)$ and $\overline{\overline{\varepsilon}}(t) = \overline{\overline{x}}(t, t_0, \overline{x}_0, \overline{\overline{x}}_0) - \overline{\overline{p}}(t, t_0, \overline{p}_0, \overline{\overline{p}}_0)$. Then accessory differential equation system takes the form

$$\begin{aligned} \frac{d\overline{\varepsilon}_s}{dt} &= Y_{s1}(t, \overline{\varepsilon}_s, \overline{\overline{\varepsilon}}_s) + Y_{1s1}(t, l_{s1}\overline{\varepsilon}_1, \ldots, l_{sq}\overline{\varepsilon}_q, l_{s1}\overline{\overline{\varepsilon}}_1, \ldots, l_{sq}\overline{\overline{\varepsilon}}_q) + \\ &+ f_{2s1}(t, e_{s1}\overline{x}_1, \ldots, e_{sq}\overline{x}_q, e_{s1}\overline{\overline{x}}_1, \ldots, e_{sq}\overline{\overline{x}}_q), \\ \frac{d\overline{\overline{\varepsilon}}_s}{dt} &= Y_{s2}(t, \overline{\varepsilon}_s, \overline{\overline{\varepsilon}}_s) + Y_{2s1}(t, l_{s1}\overline{\varepsilon}_1, \ldots, l_{sq}\overline{\varepsilon}_q, l_{s1}\overline{\overline{\varepsilon}}_1, \ldots, l_{sq}\overline{\overline{\varepsilon}}_q) + \\ &+ f_{2s2}(t, e_{s1}\overline{x}_1, \ldots, e_{sq}\overline{x}_q, e_{s1}\overline{\overline{x}}_1, \ldots, e_{sq}\overline{\overline{x}}_q), \quad s = \overline{1, q}. \end{aligned} \qquad (1_3)$$

Here

$$Y_{is}(t, \overline{\varepsilon}_s, \overline{\overline{\varepsilon}}_s) = f_{si}(t, \overline{\varepsilon}_s + \overline{p}_s(t), \overline{\overline{\varepsilon}}_s + \overline{\overline{p}}_s(t)) - f_{si}(t, \overline{p}_s(t), \overline{\overline{p}}_s(t)),$$

$$Y_{is1}(t, l_{s1}\overline{\varepsilon}_1, \ldots, l_{sq}\overline{\varepsilon}_q, l_{s1}\overline{\overline{\varepsilon}}_1, \ldots, l_{sq}\overline{\overline{\varepsilon}}_q) =$$

$$= f_{is1}(t, l_{s1}(\overline{\varepsilon}_s + \overline{p}_s(t)), \ldots, l_{sq}(\overline{\varepsilon}_s + \overline{p}_s(t)), l_{s1}(\overline{\overline{\varepsilon}}_s + \overline{\overline{p}}_s(t)), \ldots, l_{sq}(\overline{\overline{\varepsilon}}_s + \overline{\overline{p}}_s(t))) -$$

$$- f_{is1}(t, l_{s1}\overline{p}_1(t), \ldots, l_{sq}\overline{p}_q(t), l_{s1}\overline{\overline{p}}_1(t), \ldots, l_{sq}\overline{\overline{p}}_q(t)), \quad s = \overline{1, q}, \quad i = 1, 2.$$

Suppose that for the system

$$\frac{d\overline{\gamma}_s}{dt} = Y_{s1}(t, \overline{\gamma}_s, \overline{\overline{\gamma}}_s), \qquad (10)$$

$$\frac{d\overline{\overline{\gamma}}_s}{dt} = Y_{s1}(t, \overline{\gamma}_s, \overline{\overline{\gamma}}_s), \quad s = \overline{1, q},$$

there exist Lyapunov functions satisfying

$$\overline{a}_{1s}\overline{\omega}(\|\overline{\gamma}_s\|) \leqslant \overline{V}_s(t, \overline{\gamma}_s, \overline{\overline{\gamma}}_s) \leqslant \overline{a}_{2s}\overline{\omega}(\|\overline{\gamma}_s\|),$$

$$|\overline{V}_s(t,\overline{\gamma}_{s2},\overline{\overline{\gamma}}_{s2})-\overline{V}_s(t,\overline{\gamma}_{s1},\overline{\overline{\gamma}}_{s1})|\leqslant L_{s1}(\|\overline{\gamma}_{s2}-\overline{\gamma}_{s1}\|+\|\overline{\overline{\gamma}}_{s2}-\overline{\overline{\gamma}}_{s1}\|), \tag{11}$$

$$D^+\overline{V}_s(t,\overline{\gamma}_s,\overline{\overline{\gamma}}_s)|_{(10)}\leqslant -\overline{c}_s\overline{\omega}(\|\overline{\gamma}_s\|),$$

where $\overline{a}_{1s},\overline{a}_{2s},\overline{c}_s$ – positive real constants, $\overline{\omega}(\|\overline{\gamma}_s\|)$ is Khan functions, $s=\overline{1,q}$. Then

$$D^+\overline{V}_s(t,\overline{\varepsilon}_s,\overline{\overline{\varepsilon}}_s)|_{(1_3)}\leqslant \tag{12}$$

$$\leqslant -\overline{c}_s\overline{\omega}(\|\overline{\varepsilon}_s\|)+L_{s1}\sum_{j=1}^{q}M_{sj}l_{sj}\overline{\omega}(\|\overline{\varepsilon}_j\|)+L_{s1}\sum_{j=1}^{q}A_{sj}e_{sj}\overline{\omega}^2(\|\overline{\varepsilon}_j\|).$$

Here $\overline{V}_s(t,\overline{\varepsilon}_s,\overline{\overline{\varepsilon}}_s)::=\overline{V}_s(t,\overline{\gamma}_s,\overline{\overline{\gamma}}_s)$, $s=\overline{1,q}$.
Assume that there exist such $0<\overline{\rho}<1$ and q positive constants $k_1,\dots,k_q$ that

$$D^+\overline{V}_s(t,\overline{\varepsilon}_s,\overline{\overline{\varepsilon}}_s)|_{(10)}\leqslant -(1-\overline{\rho})\sum_{j=1}^{q}l_{sj}k_j\overline{\omega}(\|\overline{x}_j\|)+L_{s1}\sum_{j=1}^{q}l_{sj}A_{sj}e_{sj}\overline{\omega}^2(\|\overline{\varepsilon}_j\|),\quad s=\overline{1,q}.$$

Let us define Lyapunov function for system (1_3)

$$\overline{V}(t,\overline{\varepsilon},\overline{\overline{\varepsilon}})=\sum_{s=1}^{q}\overline{d}_s\overline{V}_s(t,\overline{\varepsilon}_s,\overline{\overline{\varepsilon}}_s),\quad where\ \ \overline{d}_s>0,\ s=\overline{1,q}.$$

From definitions of $\overline{V}_s(t,\overline{\varepsilon}_s,\overline{\overline{\varepsilon}}_s)$ and $\overline{V}_s(t,\overline{\gamma}_s,\overline{\overline{\gamma}}_s)$, $s=\overline{1,q}$, it follows that

$$\overline{\overline{a}}_1\omega(\|\overline{\varepsilon}\|)\leqslant\sum_{s=1}^{q}\overline{d}_s\overline{a}_{1s}\overline{\omega}(\|\overline{\varepsilon}_s\|)\leqslant\overline{V}(t,\overline{\varepsilon},\overline{\overline{\varepsilon}})\leqslant\sum_{s=1}^{q}\overline{d}_s\overline{a}_{2s}\overline{\omega}(\|\overline{\varepsilon}_s\|)\leqslant\overline{\overline{a}}_2\omega(\|\overline{\varepsilon}\|), \tag{13}$$

where $\overline{\overline{a}}_1$ and $\overline{\overline{a}}_2$ are positive real numbers. Hence we have the following differential inequality

$$D^+\overline{V}\leqslant(\rho-1)K(L)\overline{V}+\frac{1}{\overline{a}_1^2}\sum_{s,j=1}^{q}\overline{d}_sL_{s1}A_{sj}(E)\|\overline{x}_1\|^2, \tag{14}$$

which corresponds to the differential equation of comparison

$$D^+\overline{U}\leqslant(\rho-1)K(L)\overline{U}+\frac{1}{\overline{a}_1^2}\sum_{s,j=1}^{q}\overline{d}_sL_{s1}A_{sj}(E)\|x_1\|^2. \tag{15}$$

Herewith $\overline{V}(t_0)=\overline{V}_0=\overline{U}_0=\overline{U}(t_0)=0$, a $\overline{V}(t)\leqslant\overline{U}(t)$, $t\geqslant t_0\geqslant 0$.

$$\overline{U}=e^{(\rho-1)K(L)(t-t_0)}\overline{U}_0+\int_{t_0}^{t}e^{(\rho-1)K(L)(t-\tau)}\frac{1}{\overline{a}_1^2}\sum\overline{d}_sL_{s1}A_{sj}(E)\|x_1\|^2d\tau$$

The equation (12) has the equilibrium

$$\overline{U}=\frac{\sum_{s=1}^{q}\overline{d}_sL_{s1}A_{sj}}{(\rho-1)K(L)\overline{a}_1^2}\left[\omega^{-1}\left(\frac{1}{\overline{a}_1}\max\{\ \}\right)\right]^2,$$

then we have the estimate

$$\sup_{t\geqslant t_0\geqslant 0}|\overline{U}(t,t_0,\overline{U}_0)|\leqslant\max\left\{\frac{\sum_{s=1}^{q}\overline{d}_sL_{s1}A_{sj}}{(\rho-1)K(L)\overline{a}_1^2}[\max\{\overline{U}_0,\ \}]^2\right\}$$

Hereby, for the given definition of $\overline{V}(t,\overline{\varepsilon},\overline{\overline{\varepsilon}})$ the connective errors estimate of linearization concerning part of phase variables $\overline{x}$ takes the form

$$\sup_{t\geqslant t_0\geqslant 0}\|\overline{x}(t)\|\leqslant\frac{1}{\overline{\overline{a}}_1}\max\overline{\omega}^{-1}\left\{\frac{1}{\overline{a}_1}\max\{\ \}\right\} \tag{16}$$

Thus, we state the following theorem.

Theorem 1. *Assumptions: 1) for the system* (3) *and* (10) *there exist Lyapunov functions satisfying* (4) – (11)*. 2) the equation* (8_1) *has solutions* $0 \leqslant \overline{U}_1(\cdot) \leqslant \overline{U}_2(\cdot)$ *for* $\alpha(L) > 0$*,* $\beta(E) > 0$*. Then: 1) the solution* $y(t, t_0, y_0)$ *exist for* $t \geqslant t_0 \geqslant 0$ *and, in addition, with* $\overline{U}_0 \leqslant \overline{U}_2(\cdot)$ *satisfies the valuation* (4.6.9) *for all possible* $L \in \overline{L}$*,* $E \in \overline{E}$*; 2) differences in the systems solutions* (1_2) *and* (2_2) *with same initial data satisfy the valuation* (13) *for all* $L \in \overline{L}$*,* $E \in \overline{E}$*.*

References

1. Voronov A.A. Vvedenie v dinamiku slozhnyh upravlyaemyh sistem. Moscow: Nauka, 1985. 352 p.
2. Shchennikov A.V. Princip vklyucheniya i ustojchivopodobnye svojstva "chastichnogo"polozheniya ravnovesiya dinamicheskoj sistemy // S.-Petersburg Univ. Vestnik, 2011. Series 10., Issue 4. P. 119-132.
3. Shchennikov V.N. Ob ocenke pogreshnosti linearizacii nelinejnoj differencial'noj sistemy v kriticheskom sluchae // Diff. uravneniya, 1981. Vol. XVIII, No. 3. P. 568-571.
4. Shchennikova E.V. Postroenie konnektivnyh ocenok pogreshnostej linearizacii mnogosvyaznyh nelinejnyh sistem // Saint-Petersburg University Vestnik, 2007. Series 10, Issue 1. P. 76-83.
5. Shchennikov V.N., Shchennikova E.V. Ocenki pogreshnosti linearizacii otnositel'no chasti i vsekh fazovyh peremennyh // Differential equations, 2001. Vol. 37, Issue 1. P. 149-150.