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Analytical solutions to the FENE-P model with slip boundary conditions*

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Abstract: We study analytical solutions of equations describing steady flows of a FENE-P fluid in a channel under slip boundary conditions. The Navier slip condition and threshold-type slip conditions are considered. For the plane Poiseuille flow, we obtain explicit formulas for the velocity field, the stress in the fluid, and the configuration tensor.

Keywords: FENE-P model, polymeric fluids, Poiseuille flow, slip boundary condition, analytical solutions

1. Introduction and problem formulation

In this communication, we shall deal with the FENE-P model for dilute solutions of flexible polymer chains. This model was proposed by Peterlin [1] as a macroscopic approximation of the FENE (Finite Extensible Nonlinear Elastic) model, which is one of the most used micro-macro models in polymeric fluids [2–4]. Mathematically, the FENE-P system reads:

$$\begin{cases} \rho\left(\dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = -\nabla p + \nabla\cdot\boldsymbol{\tau} + \rho\boldsymbol{g}, \\ \nabla\cdot\boldsymbol{v} = 0, \\ \boldsymbol{\tau} = \dfrac{\eta}{\lambda}\left(\dfrac{\boldsymbol{A}}{1-\operatorname{tr}(\boldsymbol{A})/L^2} - a\mathbf{I}\right), \\ \overset{\triangledown}{\boldsymbol{A}} = -\dfrac{1}{\lambda}\left(\dfrac{\boldsymbol{A}}{1-\operatorname{tr}(\boldsymbol{A})/L^2} - a\mathbf{I}\right), \end{cases} \tag{1}$$

where ρ is the density of the fluid, $\boldsymbol{v}$ is the velocity field, p is the pressure, $\boldsymbol{\tau}$ is the extra-stress, $\boldsymbol{g}$ denotes some external forces applied to the fluid, $\boldsymbol{A}$ is the configuration tensor (this tensor is positive definite), the operator ∇ is the gradient with respect to the variables x, y, z. As usual, $\operatorname{tr}(\boldsymbol{A})$ denotes the trace of $\boldsymbol{A}$, the symbol $\overset{\triangledown}{\boldsymbol{A}}$ is used to denote the upper-convected Oldroyd derivative defined by

$$\overset{\triangledown}{\boldsymbol{A}} := \frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} - (\nabla\boldsymbol{v})\boldsymbol{A} - \boldsymbol{A}(\nabla\boldsymbol{v})^T,$$

and $\mathbf{I}$ is the identity tensor. In (1), $\eta > 0$ is polymer viscosity, $\lambda > 0$ is the relaxation time, L denotes a dimensionless parameter ($L > \sqrt{3}$), which characterizes the extensibility of polymer chains, and $a = 1/(1 - 3/L^2)$.

Sometimes it is assumed that the parameter L is sufficiently large and hence $a = 1$. However, following [5], [6], we do not accept these assumptions in our work. Note also that we consider the FENE-P model without taking into account the solvent viscosity.

The mathematical analysis of the FENE model and its approximations is rather difficult. Examples of well-posedness results for the corresponding evolution equations are presented in

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the papers [7–14]. We also refer the reader to Li and Zhang [15] and Le Bris and Lelièvre [16] for detailed mathematical overviews on micro-macro models of complex fluids.

In this paper, we are interested in findind analitical solutions for steady flows of FENE-P fluids within the space between two parallel plates ($-h \le y \le h$) under slip boundary conditions. We will use the Navier slip condition on the channel walls $y = \pm h$. This condition states that the slip velocity is directly proportional to the shear stress in the fluid (see the pioneering work of Navier [17]):

$$\begin{cases} \boldsymbol{v} \cdot \mathbf{n} = 0, \\ (\boldsymbol{\tau}\mathbf{n})_{\tan} = -k\boldsymbol{v}_{\tan}, \end{cases} \tag{2}$$

where $\mathbf{n}$ is the outer unit normal vector to the corresponding plate, k is a positive constant, and $\boldsymbol{v}_{\tan}$ denotes the component of $\boldsymbol{v}$ in the tangential direction at the channel wall, i.e.,

$$\boldsymbol{v}_{\tan} = \boldsymbol{v} - (\boldsymbol{v} \cdot \mathbf{n})\mathbf{n}.$$

We also consider threshold-type slip conditions, assuming that the slipping occurs along solid walls only when a certain threshold for the shear stress is overcome:

$$\begin{cases} \boldsymbol{v} \cdot \mathbf{n} = 0, \\ \boldsymbol{v}_{\tan} = \mathbf{0} & \text{if } \|(\boldsymbol{\tau}\mathbf{n})_{\tan}\|_{\mathbb{R}^3} \le \sigma, \\ (\boldsymbol{\tau}\mathbf{n})_{\tan} = -\big(\sigma + k\|\boldsymbol{v}_{\tan}\|_{\mathbb{R}^3}\big)\dfrac{\boldsymbol{v}_{\tan}}{\|\boldsymbol{v}_{\tan}\|_{\mathbb{R}^3}} & \text{if } \|(\boldsymbol{\tau}\mathbf{n})_{\tan}\|_{\mathbb{R}^3} > \sigma, \end{cases} \tag{3}$$

where σ is a constant threshold.

The importance of studying the effects of slip for polymer fluids is noted in many studies (see, e.g., [18–20] and the references cited therein).

It should be mentioned that analytical solutions for tube and slit flows of a FENE-P fluid (with a vanishing solvent viscosity) were first given by Oliveira [6] subject to the classical no-slip condition.

2. Finding analytical solutions

Let us assume that the steady flow in the channel is driven by constant pressure gradient

$$\frac{\partial p}{\partial x} = -\xi, \quad \xi > 0, \tag{4}$$

and $\boldsymbol{g}^T = (0, -g, 0)$, i.e., we deal with the plane Poiseuille flow.

Then for the components of the velocity $\boldsymbol{v}$, we have

$$v_x = u(y), \quad v_y = 0, \quad v_z = 0,$$

where $u = u(y)$ is an unknown function. Moreover, it can easily be checked that

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = 0, \qquad \nabla \cdot \boldsymbol{v} = 0,$$

$$\overset{\triangledown}{\boldsymbol{A}} = -(\nabla \boldsymbol{v})\boldsymbol{A} - \boldsymbol{A}(\nabla \boldsymbol{v})^T.$$

Therefore system (1) reduces to

$$\begin{cases} -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \boldsymbol{g} = 0, \\ \boldsymbol{\tau} = \dfrac{\eta}{\lambda}\left(\dfrac{\boldsymbol{A}}{1 - \operatorname{tr}(\boldsymbol{A})/L^2} - a\mathbf{I}\right), \\ (\nabla \boldsymbol{v})\boldsymbol{A} + \boldsymbol{A}(\nabla \boldsymbol{v})^T = \dfrac{1}{\lambda}\left(\dfrac{\boldsymbol{A}}{1 - \operatorname{tr}(\boldsymbol{A})/L^2} - a\mathbf{I}\right). \end{cases} \tag{5}$$

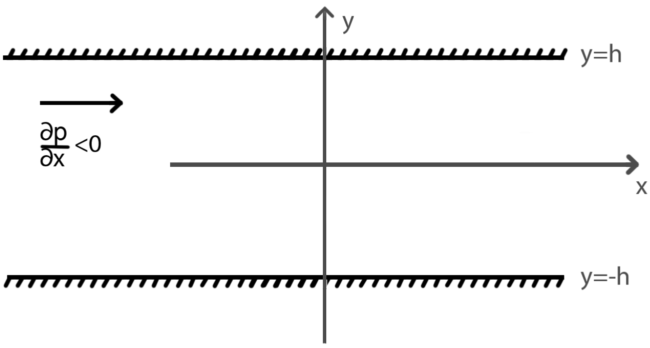

Рис. 1. Flow configuration

First, we try to eliminate the configuration tensor $\boldsymbol{A}$ from (5) and obtain a closed system with respect to $\boldsymbol{v}$ and $\boldsymbol{\tau}$.

Combining $(5)_2$ and $(5)_3$, we find

$$\boldsymbol{\tau} = \eta\big((\nabla \boldsymbol{v})\boldsymbol{A} + \boldsymbol{A}(\nabla \boldsymbol{v})^T\big). \tag{6}$$

Left-multiplying the equality $(5)_2$ by $\nabla \boldsymbol{v}$, we obtain

$$(\nabla \boldsymbol{v})\boldsymbol{\tau} = \frac{\eta}{\lambda}\left(\frac{(\nabla \boldsymbol{v})\boldsymbol{A}}{1 - \mathrm{tr}(\boldsymbol{A})/L^2} - a\nabla \boldsymbol{v}\right). \tag{7}$$

Right-multiplying $(5)_2$ by $(\nabla \boldsymbol{v})^T$, we have

$$\boldsymbol{\tau}(\nabla \boldsymbol{v})^T = \frac{\eta}{\lambda}\left(\frac{\boldsymbol{A}(\nabla \boldsymbol{v})^T}{1 - \mathrm{tr}(\boldsymbol{A})/L^2} - a(\nabla \boldsymbol{v})^T\right). \tag{8}$$

Further, summing (7) and (8), we get

$$(\nabla \boldsymbol{v})\boldsymbol{\tau} + \boldsymbol{\tau}(\nabla \boldsymbol{v})^T = \frac{\eta}{\lambda}\left(\frac{(\nabla \boldsymbol{v})\boldsymbol{A} + \boldsymbol{A}(\nabla \boldsymbol{v})^T}{1 - \mathrm{tr}(\boldsymbol{A})/L^2} - a\big(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\big)\right).$$

Taking into account (6), from the last equality we obtain

$$(\nabla \boldsymbol{v})\boldsymbol{\tau} + \boldsymbol{\tau}(\nabla \boldsymbol{v})^T = \frac{\eta}{\lambda}\left(\frac{\boldsymbol{\tau}}{\eta(1 - \mathrm{tr}(\boldsymbol{A})/L^2)} - a\big(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\big)\right). \tag{9}$$

Let us introduce a new function $\omega = \omega(y)$ defined by the following formula

$$\omega := \frac{1}{1 - \mathrm{tr}(\boldsymbol{A})/L^2}. \tag{10}$$

Multiplying (9) by λ, we arrive at

$$\lambda\left((\nabla \boldsymbol{v})\boldsymbol{\tau} + \boldsymbol{\tau}(\nabla \boldsymbol{v})^T\right) = \omega\boldsymbol{\tau} - a\eta\big(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\big). \tag{11}$$

Now, let us express ω as a function of $\mathrm{tr}(\boldsymbol{\tau})$. Take the trace of both sides of equality $(5)_2$:

$$\mathrm{tr}(\boldsymbol{\tau}) = \frac{\eta}{\lambda}\left(\frac{\mathrm{tr}(\boldsymbol{A})}{1 - \mathrm{tr}(\boldsymbol{A})/L^2} - 3a\right).$$

This yields that

$$\mathrm{tr}(\boldsymbol{A}) = \frac{L^2(\lambda\,\mathrm{tr}(\boldsymbol{\tau}) + 3a\eta)}{\eta L^2 + \lambda\,\mathrm{tr}(\boldsymbol{\tau}) + 3a\eta}. \tag{12}$$

Substituting the expression (12) into the right-hand side of (10), we obtain

$$\omega = \frac{\eta L^2 + \lambda\,\mathrm{tr}(\boldsymbol{\tau}) + 3a\eta}{\eta L^2}. \tag{13}$$

Equation (11) is equivalent to the following system

$$\begin{cases} 2\lambda\dfrac{du}{dy}\tau_{xy} - \omega\tau_{xx} = 0, \\ \lambda\dfrac{du}{dy}\tau_{yy} - \omega\tau_{xy} + a\eta\dfrac{du}{dy} = 0, \\ \lambda\dfrac{du}{dy}\tau_{yz} - \omega\tau_{xz} = 0, \\ \omega\tau_{yy} = 0, \\ \omega\tau_{yz} = 0, \\ \omega\tau_{zz} = 0. \end{cases} \tag{14}$$

Taking into account (10), we obviously have $\omega(y) \neq 0$ for any y such that $-h \leq y \leq h$. Therefore, from $(14)_4$, $(14)_5$, $(14)_6$ it follows that

$$\tau_{yy} = \tau_{yz} = \tau_{zz} = 0.$$

In addition, if we combine this with $(14)_2$ and $(14)_3$, we get

$$-\omega\tau_{xy} + a\eta\frac{du}{dy} = 0, \tag{15}$$

$$\tau_{xz} = 0.$$

Now, multiply equality $(14)_1$ by $-a\eta/\omega$, equality (15) by $2\lambda\tau_{xy}/\omega$ and add the results; this gives

$$\tau_{xx} = \frac{2\lambda}{a\eta}\tau_{xy}^2. \tag{16}$$

It follows from (4) and $(5)_1$ that

$$\frac{\partial\tau_{xy}}{\partial y} = \frac{\partial p}{\partial x} = -\xi,$$

whence

$$\tau_{xy} = -\xi y. \tag{17}$$

Substituting the value of τ_{xy} into the right-hand side of (16), we obtain

$$\tau_{xx} = \frac{2\lambda\xi^2 y^2}{a\eta}. \tag{18}$$

Thus, we have

$$\boldsymbol{\tau} = \begin{bmatrix} \dfrac{2\lambda\xi^2 y^2}{a\eta} & -\xi y & 0 \\ -\xi y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{19}$$

Now we turn to finding of the configuration tensor $\boldsymbol{A}$. From $(5)_2$ it follows that

$$\boldsymbol{A} = \frac{1}{\omega}\left(\frac{\lambda}{\eta}\boldsymbol{\tau} + a\mathbf{I}\right),$$

whence, taking into account (19) and (13), we get

$$\boldsymbol{A} = \begin{bmatrix} \dfrac{L^2(2\xi^2\lambda^2y^2 + a^2\eta^2)}{2\xi^2\lambda^2y^2 + aL^2\eta^2 + 3a^2\eta^2} & -\dfrac{a\eta L^2\lambda\xi y}{2\xi^2\lambda^2y^2 + aL^2\eta^2 + 3a^2\eta^2} & 0 \\ -\dfrac{a\eta L^2\lambda\xi y}{2\xi^2\lambda^2y^2 + aL^2\eta^2 + 3a^2\eta^2} & \dfrac{a^2\eta^2L^2}{2\xi^2\lambda^2y^2 + aL^2\eta^2 + 3a^2\eta^2} & 0 \\ 0 & 0 & \dfrac{a^2\eta^2L^2}{2\xi^2\lambda^2y^2 + aL^2\eta^2 + 3a^2\eta^2} \end{bmatrix}$$

Applying Sylvester's criterion, we see that the tensor $\boldsymbol{A}$ is positive definite. This confirms the correctness of the solution obtained here.

To conclude, we must find the velocity u. Using (13), (17), and (18), from (15) we find the velocity gradient:

$$\frac{du}{dy} = -\frac{\xi(2\lambda^2\xi^2y^2 + a\eta^2L^2 + 3a^2\eta^2)y}{a^2\eta^3L^2}.$$

Integrating the last equality with respect to y, we get

$$u(y) = -\frac{\xi\big(\xi^2\lambda^2y^4 + (aL^2 + 3a^2)\eta^2y^2\big)}{2a^2L^2\eta^3} + C,$$

where C is a constant.

In accordance with the Navier slip boundary condition (2), the following equality must be satisfied:

$$\xi h = ku(\pm h),$$

or equivalently,

$$\xi h = -\frac{k\xi(\xi^2\lambda^2h^4 + (aL^2 + 3a^2)\eta^2h^2)}{2a^2L^2\eta^3} + kC,$$

whence

$$C = \frac{\xi h}{k} + \frac{\xi\big(\xi^2\lambda^2h^4 + (aL^2 + 3a^2)\eta^2h^2\big)}{2a^2L^2\eta^3}.$$

Thus, we have

$$u(y) = \frac{\xi^3\lambda^2(h^4 - y^4) + (aL^2 + 3a^2)\xi\eta^2(h^2 - y^2)}{2a^2L^2\eta^3} + \frac{\xi h}{k}. \tag{20}$$

Under the threshold slip boundary condition (3), the velocity u is determined by the following formulas:

$$u(y) = \begin{cases} \dfrac{\xi^3\lambda^2(h^4 - y^4) + (aL^2 + 3a^2)\xi\eta^2(h^2 - y^2)}{2a^2L^2\eta^3} & \text{if } \xi h \le \sigma, \\ \dfrac{\xi^3\lambda^2(h^4 - y^4) + (aL^2 + 3a^2)\xi\eta^2(h^2 - y^2)}{2a^2L^2\eta^3} + \dfrac{\xi h - \sigma}{k} & \text{if } \xi h > \sigma. \end{cases} \tag{21}$$

It is readily seen that (21) reduces to (20) as $\sigma \to 0$.

3. Concluding remarks

Note that the velocity field

$$u_0(y) = \frac{\xi^3\lambda^2(h^4 - y^4) + (aL^2 + 3a^2)\xi\eta^2(h^2 - y^2)}{2a^2L^2\eta^3}.$$

corresponds to the no-slip condition on the channel walls.

We can rewrite (21) as follows

$$u(y) = \begin{cases} u_0(y) & \text{if } \xi h \le \sigma, \\ u_0(y) + \dfrac{\xi h - \sigma}{k} & \text{if } \xi h > \sigma. \end{cases}$$

Clearly, ξh is one of the key parameters for the problem under consideration. If ξh overcomes the threshold value σ, then the slip regime holds at solid surfaces, otherwise the fluid adheres to the channel walls.

In the case $L \to +\infty$ (infinite extensibility), we obtain the velocity solution

$$u(y) = \begin{cases} \dfrac{\xi}{2\eta}(h^2 - y^2) & \text{if } \xi h \le \sigma, \\ \dfrac{\xi}{2\eta}(h^2 - y^2) + \dfrac{\xi h - \sigma}{k} & \text{if } \xi h > \sigma, \end{cases}$$

which is parabolic as for Newtonian fluids. However, the constitutive law reduces to the upper convected Maxwell model (see [6] for more detail).

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