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About global and local orthogonal splines

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Аннотация: The procedures of orthogonalization of splines generating global or local orthogonal splines are considered. It is noted that many existing orthogonalization procedures for the formation of local orthogonal splines make it possible to minimize sizes growth of splines compact supports, but initial sizes of compact supports increases during orthogonalization. The importance of Leontiev theory of splines, based on obtaining local orthogonal splines using the orthogonalization procedure that preserves sizes of compact supports of splines, is emphasized.

Ключевые слова: basis of Hilbert space, Haar functions, Faber functions, Franklin functions, Schoenberg splines, global or local orthogonal splines, orthogonalization procedure preserving compact supports of functions, Gram-Schmidt orthogonalization procedure.

1. Comparison of different global and local orthogonal splines

The Haar system of functions [1], which appeared in 1910, makes it possible to approximate functions in the space of continuous functions. This system is not a basis of a given space, but is historically the first basis of the Hilbert space of quadratically summable functions. On the grid that appears at each current step of the halving the corresponding part of the system of Haar functions is equivalent to a system of step functions. Such a finite set of Haar functions and a grid system of step functions are associated with B-splines of degree zero, which are local and orthogonal, but have discontinuities of the first kind. Integrating the Haar functions led in 1910 to the Faber system of functions [2], which is historically the first basis of the space of continuous functions. On each particular grid, Faber functions are associated with first degree B-splines, which are continuous local splines but not orthogonal. The application of the Gram-Schmidt orthogonalization procedure to the Faber system of functions gave rise in 1928 to the basis system of Franklin functions [3], which are global orthogonal continuous splines that lost their compact supports during the transition from Faber functions. The theory of splines begins with the article [4], in which cubic Schoenberg splines were constructed, which are local and continuous, but not orthogonal.

The Gram-Schmidt orthogonalization procedure for systems of compactly supported functions leads to an increase in the sizes of compact supports of functions. The compactness of the supports of the resulting functions and their orthogonality are incompatible in the case of this procedure. A single volume 5 of the journal Numerical Algorithms with numbers 1-4 was published in November 1993 and contains the article [5] in which global piecewise linear continuous splines obtained on the basis of B-splines using a simplified Gram-Schmidt orthogonalization procedure were proposed. The article [6] applies global splines [5] and notes that they are constructed using the Gram-Schmidt orthogonalization procedure. In the case of B-splines of the first degree (Faber functions), the orthogonalization procedure used in the article [5] is a special case of the general Gram-Schmidt procedure used to construct the orthogonal Franklin basis [3], and therefore does not seem to possess fundamental scientific novelty. In [5], orthogonal global piecewise linear splines are considered, just as in the case of the Franklin function system, but unlike [3] they are connected to only one grid. The expansion of domains of the compact supports at each step of applying the simplified orthogonalization formula [5] is noted in the article [5] and it is recognized that global splines are proposed in this article. The article [5] proposes a procedure for forming a system of global splines based on local B-splines of the first degree, in which only neighboring splines on a given grid are orthogonal. The article [5] also provides the algorithm for forming a system of global piecewise cubic splines in which only neighboring splines are orthogonal on a given grid. Therefore, all the comments presented above regarding piecewise linear splines [5] also apply to piecewise cubic splines [5].

In May 1993 [7], a generalization of first degree B-splines (Courant functions) was given for the first time without changing their compact supports, leading to orthogonal local continuous piecewise linear B-splines. In January 1994, such a generalization of B-splines was presented in more details [8]. The list of references in the monograph [9] contains information about articles in which, in subsequent years, the idea of forming generalized local continuous orthogonal splines using the orthogonalization procedure [7,8] preserving the compact supports of the original functions, was developed and justified. The orthogonalization algorithm [7,8,9] applied to cubic Schoenberg splines, while preserving their compact supports and using auxiliary step functions, is described in article [10], and in article [11] it is developed using four auxiliary cubic Schoenberg splines.

In 1988, the orthogonal basis of Daubechies wavelets with compact supports was created, but such wavelets are constructed approximately using an iterative procedure, have low smoothness, are not expressed in analytical form, and do not possess symmetry properties, as noted in [9]. The article [12] suggests a procedure for constructing orthogonal scaling functions with compact supports. Examples of such scaling functions and their corresponding wavelets are given [12]. The procedure [12] is similar to the Daubechies procedure and generates orthogonal wavelets with compact supports having low smoothness. The article [13] discusses the symmetric Gram-Schmidt procedure with a limitation, when applied, of the growth in the sizes of compact supports inherent in the general Gram-Schmidt procedure. The work [14] is devoted to algorithms for constructing orthogonal splines while minimizing the sizes of their compact supports, that is, splines approaching local splines in their properties, using various procedures, in particular, using the Gram-Schmidt procedure. At the same time, the articles [12–14] do not take into account the fact that the theory of orthogonal local continuous splines was built on the basis of another orthogonalization procedure that preserves compact supports of functions and does not require minimizing their sizes during orthogonalization, starting in 1993 [7, 8]. This theory is presented in the monograph [9] and it continues to evolve. In the following years after 1993, many new results were obtained within the framework of this theory, for example, [10, 11]. The works [5–14] show that scientific directions related to the creation of grid bases consisting of global or local orthogonal splines and their applications in computational mathematics are relevant.

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