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Cash Management Using ODE-Based Modeling

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Abstract: In this short communication paper a continuous-time model for dynamic corporate cash management is presented, utilizing the principles of optimal control theory. The evolution of a company's liquidity position is formulated through a system of Ordinary Differential Equations (ODEs) that correctly distinguishes between stocks (cash, debt, investments) and flows (rates of borrowing and investment). The problem is framed to minimize financing costs and maximize investment returns over a finite horizon. By applying Pontryagin's Maximum Principle, the necessary conditions for optimal financial interventions are derived. The resulting control policies are threshold-based strategies, dynamically guided by the economic shadow prices of cash, debt, and invested capital. This framework provides a financially intuitive and mathematically sound foundation for proactive liquidity management, bridging theoretical control theory with practical corporate finance.

Keywords: ordinary differential equations, cash management, optimal control.

1. Introduction

Effective liquidity management is paramount for corporate stability and growth. A firm must continuously balance its cash reserves to cover operational expenses, mitigate risks, and seize investment opportunities. An optimized cash balance is crucial for covering operational costs, financing investments, and hedging against uncertainties. Traditional, discrete-time financial models often fail to capture the high-frequency dynamics of cash flows, limiting their utility to historical analysis or long-term forecasting rather than real-time decision support. These methods primarily offer historical snapshots, serving as reactive or forecasting tools rather than real-time control instruments.

To address this gap, a continuous-time modeling approach is proposed, utilizing a system of Ordinary Differential Equations (ODEs). This allows for a granular representation of the interplay between operational cash flows, short-term borrowing, and investment activities. The central challenge lies in determining the optimal timing and magnitude of these financial interventions to achieve a specific corporate objective, such as minimizing costs or maximizing terminal value. This shifts financial management from reactive to proactive, providing a strategic advantage in dynamic economic environments.

This problem is naturally formulated as one of optimal control. By defining the company's financial state with a set of state variables and its actions (borrowing, investing) as control variables, a policy can be sought that optimizes a predefined objective functional. Pontryagin's Maximum Principle provides the mathematical machinery to solve such problems, yielding a set of necessary conditions that the optimal policy must satisfy. Thus, an ODE-based model is proposed for a company's daily cash balance.

2. The Dynamic System Model

To accurately model the system, a clear distinction between stocks and flows is essential. The financial state of the company at any given time t is comprehensively

described by three state variables:
- $C(t)$: The cash balance.
- $B(t)$: The total outstanding short-term debt.
- $I(t)$: The total capital allocated to short-term investments.

The company can influence these states through four control variables:
- $u_B(t) \geq 0$: The rate of new borrowing.
- $u_P(t) \geq 0$: The rate of debt repayment.
- $u_I(t) \geq 0$: The rate of new investment.
- $u_S(t) \geq 0$: The rate of selling/liquidating investments.

The system's dynamics are governed by the following system of ODEs:

$$\frac{dC}{dt} = R(t) - E(t) - r_B B(t) + r_I I(t) + u_B(t) - u_P(t) - u_I(t) + u_S(t) \tag{1}$$

$$\frac{dB}{dt} = u_B(t) - u_P(t) \tag{2}$$

$$\frac{dI}{dt} = u_I(t) - u_S(t) \tag{3}$$

where:
- $R(t)$ and $E(t)$ are the exogenous rates of revenue and expenses.
- r_B is the interest rate on outstanding debt (a cost).
- r_I is the rate of return on invested capital (a gain).

This formulation correctly applies interest rates r_B and r_I to the stocks of debt $B(t)$ and investments $I(t)$, respectively. The controls are subject to constraints, such as non-negativity and upper bounds, and the state variables must also satisfy constraints (e.g., $C(t) \geq 0$, $B(t) \geq 0$, $I(t) \geq 0$).

3. The Optimal Control Problem

The objective is to manage liquidity over a finite horizon $[0, T]$ to maximize the terminal value of the firm, which we can equate to maximizing net assets. This is equivalent to minimizing a cost functional J:

$$J = \int_0^T \left(r_B B(t) - r_I I(t)\right) dt - C(T) \tag{4}$$

The integrand $L = r_B B(t) - r_I I(t)$ represents the net financing cost at time t. The terminal cost $\Phi(C(T)) = -C(T)$ ensures the objective is to maximize the final cash balance, assuming terminal debt and investments can be settled.

4. Application of the Pontryagin's Maximum Principle

To derive the optimal control policy, the Hamiltonian H is defined:

$$H = L + \lambda_C \frac{dC}{dt} + \lambda_B \frac{dB}{dt} + \lambda_I \frac{dI}{dt} \tag{5}$$

where $\lambda_C(t)$, $\lambda_B(t)$, and $\lambda_I(t)$ are the adjoint variables (or shadow prices) associated with cash, debt, and investments.

Substituting the dynamics from Equations (1)-(3) and the cost function L:

$$H = (r_B B(t) - r_I I(t)) + \lambda_C (R(t) - E(t) - r_B B(t) + r_I I(t) + u_B - u_P - u_I + u_S) + \lambda_B (u_B - u_P) + \lambda_I (u_I - u_S). \quad (6)$$

The adjoint equations are given by $\dot{\lambda} = -\frac{\partial H}{\partial x}$ for each state x:

$$\frac{d\lambda_C}{dt} = -\frac{\partial H}{\partial C} = 0 \implies \lambda_C(t) = \text{const}, \quad (7)$$

$$\frac{d\lambda_B}{dt} = -\frac{\partial H}{\partial B} = -(r_B - \lambda_C r_B) = -r_B(1 - \lambda_C), \quad (8)$$

$$\frac{d\lambda_I}{dt} = -\frac{\partial H}{\partial I} = -(-r_I + \lambda_C r_I) = r_I(1 - \lambda_C), \quad (9)$$

with the transversality conditions at time T:

$$\lambda_C(T) = -\frac{\partial \Phi}{\partial C(T)} = -(-1) = 1,$$

$$\lambda_B(T) = -\frac{\partial \Phi}{\partial B(T)} = 0,$$

$$\lambda_I(T) = -\frac{\partial \Phi}{\partial I(T)} = 0.$$

From $\lambda_C(T) = 1$ and $\dot{\lambda}_C = 0$, we have $\lambda_C(t) = 1$ for all $t \in [0, T]$. This simplifies the other adjoint equations: $\dot{\lambda}_B = 0$ and $\dot{\lambda}_I = 0$. With their terminal conditions, we get $\lambda_B(t) = 0$ and $\lambda_I(t) = 0$. This result is an artifact of the simplified cost function. A more complex model with transaction costs or non-linear rates would yield dynamic shadow prices.

Let's re-group the Hamiltonian by controls to find the optimal policies:

$$H = \ldots + (\lambda_C + \lambda_B)u_B - (\lambda_C + \lambda_B)u_P + (\lambda_I - \lambda_C)u_I - (\lambda_I - \lambda_C)u_S \quad (10)$$

To minimize H, the controls must be chosen based on the signs of their respective coefficients (the switching functions):

- **Borrowing (u_B):** The coefficient is $(\lambda_C + \lambda_B)$. To minimize H, if $\lambda_C + \lambda_B < 0$, means borrowing at the maximum rate. If $\lambda_C + \lambda_B > 0$, means refraining from borrowing ($u_B = 0$). This means borrowing only if the marginal value of cash $(-\lambda_C)$ is greater than the marginal cost of debt (λ_B).

- **Repayment (u_P):** The coefficient is $-(\lambda_C + \lambda_B)$. To minimize H, if $-(\lambda_C + \lambda_B) < 0$, means repayment at the maximum rate. This is the opposite condition to borrowing.

- **Investment (u_I):** The coefficient is $(\lambda_I - \lambda_C)$. Investment $(u_I > 0)$ if $\lambda_I - \lambda_C < 0$, i.e., if $\lambda_C > \lambda_I$. This means investment only if the marginal value of cash is greater than the marginal value of the investment. This is financially logical: cash is converted to an investment if the investment is "cheaper"(has a lower shadow price) than cash.

- **Liquidation (u_S):** The coefficient is $-(\lambda_I - \lambda_C)$. Investment liquidation $(u_S > 0)$ if $-(\lambda_I - \lambda_C) < 0$, i.e., if $\lambda_I > \lambda_C$. Investments should be converted back into cash when their value exceeds that of holding cash.

5. Conclusion

This short communication paper demonstrates the utility of an ODE-based optimal control framework for dynamic cash management via formulation of a mathematically sound and financially intuitive optimal control model for corporate cash management. By accurately defining the state dynamics and objective function, control policies that align with fundamental economic principles are derived. The framework demonstrates that decisions regarding borrowing, investment, and repayment should be governed by the relative shadow prices of cash, debt, and invested capital. Pontryagin's Maximum Principle yields necessary conditions for these policies, often leading to dynamic threshold strategies, providing a robust foundation for corporate financial engineering.

While this deterministic model provides a strong foundation, future work should incorporate stochastic elements for revenues, expenses, and market rates. This would lead to a stochastic optimal control problem, likely requiring solution via Hamilton-Jacobi-Bellman (HJB) equations. Nonetheless, the presented ordinary differential equation (ODE)-based framework serves as a crucial first step in building quantitative, proactive tools for corporate financial engineering. Future work could integrate this model with broader corporate finance decisions (e.g., capital budgeting, dividend policy) for a holistic financial management framework.

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