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Linear inverse boundary value problem for equation of small transverse vibrations of a rod in the environment with resistance and integral condition of the first kind

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Abstract: In the present paper, a linear inverse boundary value problem with nonlocal integral conditions for equation of the small transverse vibrations of a rod in the environment with resistance is investigated.

Keywords: inverse problem, classical solution, nonlinear integral equation, integro-differential equation.

There are many cases where the needs of the practice bring about the problems of Determining coefficients or the right hand side of differential equations from some knowledge of its solutions. Such problems are called inverse boundary value problems of mathematical physics. Inverse boundary value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control in industry etc., which makes them an active field of contemporary mathematics. Currently, problems with non-local conditions for partial differential equations are of great interest, which is due to the need to generalize the classical problems of mathematical physics in connection with the mathematical modeling of a number of physical processes studied by modern science [1]. Among non-local problems, of great interest are problems with integral conditions. Nonlocal integral conditions describe the behavior of the solution at interior points of the domain in the form of some mean. Examples include problems arising from the study of diffusion of particles in a turbulent plasma [1], the processes of heat propagation [2, 3] of the process of moisture transfer from capillary-simple media [4], as well as the study of some inverse problems of mathematical physics. In [5], a problem with nonlocal in time integral conditions for a hyperbolic equation was considered. In [6], an inverse boundary value problem with an unknown time-dependent coefficient was investigated for a fourth-order Boussinesq equation with nonlocal second-order integral in time.

Consider the equation [7]

$$u_{tt}(x,t) + 2\alpha u_t(x,t) + \gamma u_{xxxx}(x,t) = a(t)g(x,t) + f(x,t) \tag{1}$$

in the domain $D_T = \{(x,t) : 0 \leq x \leq 1,\ 0 \leq t \leq T\}$ in the domain of an inverse boundary value problem with initial conditions

$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x)\ (0 \leq x \leq 1), \tag{2}$$

the boundary conditions

$$u_x(0,t) = u_x(1,t) = u_{xxx}(0,t) = 0\ (0 \leq t \leq T), \tag{3}$$

nonlocal integral condition

$$\int\limits_0^1 u(x,t)dx = 0\ (0 \leq t \leq T) \tag{4}$$

and with the additional conditions

$$u(0,t) = h(t)\ (0 \le t \le T) \tag{5}$$

where $\alpha > 0, \gamma > 0$- given numbers, ,$f(x,t)$, $g(x,t)$, $\phi(x)$,$\psi(x)$, $h(t)$, – given functions, $u(x,t)$and $a(t)$- desired functions.

Denote

$$\tilde{C}^{4,2}(D_T) = \left\{ u(x,t) :\ u(x,t) \in C^{2,2}(D_T), u_{xxxx}(x,t) \in C(D_T) \right\}.$$

Definition 1. *The classical solution of the inverse boundary value problem* (1)-(4) *is the pair* $\{u(x,t),\ a(t)\}$ *of functions* $u(x,t) \in \tilde{C}^{4,2}(D_T)$ *and* $a(t) \in C[0,T]$ *satisfying equation (1) in* D_T, *condition* (2) *in* $[0,1]$ *and conditions* (3)-(4) *in* $[0,T]$.

The following theorem is true.

Theorem 2. *Assume that* $f(x,t), g(x,t) \in C(D_T)$, $\phi(x)$, $\psi(x) \in C[0,\ 1]$, $h(t) \in C^2[0,T]$ $g(0,t) \ne 0\ (0 \le t \le T)$, *and the compatibility conditions*

$$\int_0^1 \phi(x)dx = 0,\ \int_0^1 \psi(x)dx = 0,\ \phi(0) = h(0),\ \psi 0 = h'(0)$$

holds. Then the problem of finding a classical solution of (1)-(5) is equivalent to the problem of determining the functions $u(x,t) \in \widetilde{C}^{4,2}(D_T)$ *and* $a(t) \in C[0,T]$, *satisfying the conditions* (1)-(3) *and the relation*

$$u_{xxx}(1,t) = 0\ (0 \le t \le T), \tag{6}$$

$$h''(t) + 2\alpha h'(t) + \gamma u_{xxxx}(0,t) = a(t)g(0,t) + f(0,t) \tag{7}$$

After applying the formal scheme of the Fourier method, finding the first component $u(x,t)$ of any solution $\{u(x,t),\ a(t)\}$ to problem (1)-(3), (6), (7) reduces to solving the following integro-differential equation:

$$u(x,t) = \phi_0 + \frac{1}{2\alpha}(1 - e^{-2\alpha t})\psi_0 + \frac{1}{2\alpha}\int_0^1 (1 - e^{-2\alpha(1-t)})F_0(\tau;a)d\tau +$$
$$+ \sum_{k=1}^{\infty} \left\{ e^{-\alpha t}\left[\varphi_k \left(\cos\beta_k t + \frac{\alpha}{\beta_k}\sin\beta_k t\right) + \frac{1}{\beta_k}\psi_k \sin\beta_k t\right] + \right.$$
$$\left. + \frac{1}{\beta_k - \alpha}\int_0^t e^{-\alpha(t-\tau)}F_k(\tau;a)\sin\beta_k(t-\tau)d\tau \right\}\cos\lambda_k x. \tag{8}$$

where

$$F_k(t;a) = a(t)g_k(t) + f_k(t),\ f_k(t) = m_k\int_0^1 f(x,t)\cos\lambda_k x dx\ ,\ g_k(t) = m_k\int_0^1 g(x,t)\cos\lambda_k x dx\ ,$$

$$\varphi_k = m_k\int_0^1 \varphi(x)\cos\lambda_k x dx\ ,\ \psi_k = m_k\int_0^1 \psi(x)\cos\lambda_k x dx,\ \lambda_k = k\pi\ (k = 0,1,2,...),$$

$$\beta_k = \sqrt{\gamma\lambda_k^4 - \alpha}\ (k = 1, 2, ...),\ \pi^4\gamma - \alpha > 0,\ m_k = \begin{cases} 1, & k = 0, \\ 2 & k = 1, 2, ... \end{cases}.$$

Further, using equation (8), from condition (7) to determine the second component of any solution $\{u(x,t),\ a(t)\}$ of problem (1)-(3), (6), (7) we obtain the following nonlinear integral equation:

$$a(t) = [g(0,t]^{-1}\{h''(t) + 2\alpha h'(t) - f(0,t) + \\ + \gamma\sum_{k=1}^{\infty}\lambda_k^4\left[e^{-\alpha t}\left(\varphi_k\left(\cos\beta_k t + \frac{\alpha}{\beta_k}\sin\beta_k t\right) + \frac{1}{\beta_k}\psi_k\sin\beta_k t\right)\right. \\ \left. + \frac{1}{\beta_k - \alpha}\int_0^t e^{-\alpha(t-\tau)}F_k\ (\tau; a)\sin\beta_k(t-\tau)d\tau]\right\}. \quad (9)$$

Thus, the solution of problem (1)-(3), (6), (7) was reduced to the solution of system (8),(9), with respect to the unknown functions $u(x,t)$and $a(t)$.

The following Theorems hold:

Theorem 3. *Let us assume that the given data conditions (1)-(3), (6),(7) satisfy the following relations :*

1. $\phi(x) \in C^4[0,1],\ \phi^{(5)}(x) \in L_2(0,1)$ *и* $\phi'(0) = \phi'(1) = \phi'''(0) = \phi'''(1) = 0$;

2. $\psi(x) \in C^2[0,1],\ \psi'''(x) \in L_2(0,1)$ *и* $\psi'(0) = \psi'(1) = 0$;

3. $f(x,t), f_x(x,t), f_{xx}(x,t) \in C(D_T),\ f_{xxx}(x,t) \in L_2(D_T)$ *and* $f_x(0,t) = f_x(1,t) = 0\ (0 \le t \le T)$*;*

4. $g(x,t), g_x(x,t), g_{xx}(x,t) \in C(D_T),\ g_{xxx}(x,t) \in L_2(D_T)$ *and* $g_x(0,t) = g_x(1,t) = 0\ , g(0,t) \ne 0\ (0 \le t \le T)$*;*

5. $h(t) \in C^2[0,T]\ \alpha > 0, \gamma > 0,\ \pi^4\gamma - \alpha > 0$.

Then there exist the unique solution $u(x,t) \in \widetilde{C}^{4,2}(D_T)a(t) \in C[0,T]$ *of the problem (1)-(3),(6),(7).*

Theorem 4. *Let all conditions of Theorem 2 fulfilled ,* $\int_0^1 f(x,t)dx = 0$ *,* $\int_0^1 g(x,t)dx = 0\ (0 \le t \le T)$ *,* $\int_0^1 \phi(x)dx = 0,\ \int_0^1 \psi(x)dx = 0,\ \phi(0) = h(0),\ \psi 0 = h'(0)$ *.*

Then there exists the unique classic solution $u(x,t) \in \widetilde{C}^{4,2}(D_T)$*,* $a(t) \in C[0,T]$ *of the problem* (1)-(5).

References

1. Samara A. A. On some problems of the theory of differential equations // Differential Equations. 1980. Vol. 16, no. 11. P. 1925-1935.

2. Cannon J. R. The solution of energy equation to the energy of Quart // Applied Mathematics. 1963. Vol. 5, no. 21. P. 1555-1560.

3. Ionkin N. I. The solution of a boundary value problem of the theory of heat conduction with a nonclassical boundary condition // Differential Equations. 1977. Vol. 13, no. 2. P. 294-304.

4. Nakhushev A. M. About one approximate method for solving boundary value problems for differential equations and its approximation to the dynamics of soil moisture and groundwater // Differential Equations. 1982. Vol. 18, no. 1. P. 72-81.

5. Kirichenko S. V. On a boundary value problem with nonlocal in time conditions for a one-dimensional hyperbolic equation // Bulletin of Samara University. Natural Science Series. 2013. No. 6(107). P. 31-39.

6. Megraliev Ya. T., Alizade F. Kh. An inverse boundary value problem for a fourth-order Boussinesq equation with nonlocal integral conditions of the second kind // Bulletin of Udmurt University. Mathematics. Mechanics. Computer Science. 2016. Vol. 26, no. 4. P. 503-514.

7. Budak B. M., Samarskii A. A., Tikhonov A. N. Collection of Problems in Mathematical Physics. Moscow: Nauka, 1972. 688 p.