MSC 35A01, 35A02, 35A09, 35B45, 35L20, 35L71

Classical Solution of the First Mixed Problem for the Liouville Equation in a Half-Strip

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Abstract: We consider a mixed problem for a nonlinear hyperbolic Liouville equation in a half-strip of the plane. We construct the solution in an implicit analytical form as a solution to some coupled integral equations. We prove the solvability of these integral equations using the Leray-Schauder theorem. We derive the corresponding a priori estimate by energy methods. For the problem under consideration, the uniqueness of the solution is proved, and conditions are established under which its classical solution exists.

Keywords: mixed problem, Liouville equation, hyperbolic equation, classical solution, matching conditions.

1. Statement of the problem

In the domain $Q = (0,\infty) \times (0,l)$, where $l > 0$, of two independent variables $(t,x) \in Q \subset \mathbb{R}^2$, we consider the nonlinear equation

$$\partial_{tt}u(t,x) - a^2\partial_{xx}u(t,x) + \lambda_1(t,x)\exp(\lambda_2 u(t,x)) = 0, \tag{1}$$

where

$$\frac{\lambda_1(t,x)}{\lambda_2} \geq 0, \quad \frac{\partial_t \lambda_1(t,x)}{\lambda_2} \leq 0 \quad (t,x) \in \overline{Q}. \tag{2}$$

Equation (1) is equipped with the initial conditions

$$u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \psi(x), \quad x \in [0,l], \tag{3}$$

and the boundary condition

$$u(t,0) = u(t,l) = 0, \quad t \in [0,\infty), \tag{4}$$

where φ and ψ are functions given on the range $[0,l]$.

2. Main results

We formulate the main results as follows.

Theorem 1. *Let the smoothness conditions $\varphi \in C^2([0,l])$, $\psi \in C^2([0,l])$, and $\lambda_1 \in C^1(\overline{Q})$ be satisfied. The first mixed problem (1)–(4) has a unique solution u in the class $C^2(\overline{Q})$ if and only if the matching conditions*

$$\varphi(0) = \varphi(l) = 0, \tag{5}$$

$$\psi(0) = \psi(l) = 0, \tag{6}$$

$$a^2\varphi''(0) - \lambda_1(0,0)\exp(\lambda_2\varphi(0)) = 0, \quad (7)$$

$$a^2\varphi''(l) - \lambda_1(0,l)\exp(\lambda_2\varphi(l)) = 0. \quad (8)$$

are satisfied.

P r o o f. The following is the sketch of the proof.

1. Using the energy method, we derive the following apriori estimate

$$\|u\|_{C(\overline{Q})} \leq C\big(\|\psi\|_{L^2([0,l])} + \|\varphi'\|_{L^2([0,l])} + \|\lambda_1(0,\cdot)\exp(\lambda_2\varphi(\cdot))\|^{1/2}_{L^1([0,l])}\big), \quad (9)$$

where C is some constant depending only on the numbers l, a and λ_2.

2. We prove that under the smoothness conditions $\varphi \in C^2([0,l])$, $\psi \in C^2([0,l])$, and $\lambda_1 \in C^1(\overline{Q})$ the solution $u \in C^2(\overline{Q^{(0)}})$ of problem (1)–(4) is equivalent to a continuous solution of the integral equation

$$u(t,x) = K[u](t,x), \quad (t,x) \in \overline{Q^{(0)}}, \quad (10)$$

where

$$Q^{(i)} = \left(\frac{il}{2a}, \frac{(i+1)l}{2a}\right) \times (0,l), \quad i = 0,1,\ldots$$

and $K\colon C(\overline{Q^{(0)}}) \mapsto C(\overline{Q^{(0)}})$ is a compact continuous operator, if and only if conditions (5)–(8) are satisfied.

3. Using estimate (9), we solve Eq. (10) by the Leray–Schauder fixed point theorem.

4. We prove the uniqueness of a solution to Eq. (10) in the class $C(\overline{Q^{(0)}})$ by contradiction.

5. We extend the constructed solution to problem (1)–(4) sequentially and uniquely to the sets $\overline{Q^{(i)}}$, $i = 1,2,\ldots$ using the conjugation condition. Since $\bigcup\limits_{j=0}^{\infty} \overline{Q^{(j)}} = \overline{Q}$, we obtain a unique solution to problem (1)–(4) on the set $\overline{Q}$.

T h e p r o o f i s f i n i s h e d.

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