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Convergence of Fourier Method connected with Orthogonal Splines

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Abstract: Fourier series, Fourier method and spline approximations have wide scopes. The generalized Fourier method associated with the use of finite Fourier series and orthogonal splines was applied earlier for solving the parabolic initial boundary value problems for regions with curvilinear boundaries. Here another analogous generalized Fourier method is applied for solving parabolic initial boundary value problems in noncanonical regions and the investigation of convergence of this method is proposed. The study of convergence is based on the theory of finite difference methods. This method gives solutions in form of finite Fourier series which structure is similar to that of partial sums of an infinite Fourier series of an exact solution. As a number of grid nodes increases in a region, a finite Fourier series approach an exact solution of a parabolic initial boundary value problem. The investigation of convergence shows efficiency of the algorithm of the generalized Fourier method in solving parabolic initial boundary value problems for noncanonical regions. This method yields the approximate analytical solutions in the form of the sequence of finite Fourier series. The use of orthogonal splines brings together numerical and analytical methods – finite difference methods and the Fourier method, expanding the scope of their applications.

Keywords: parabolic initial boundary value problems, noncanonical regions, curvilinear boundary, the method of separation of variables, finite Fourier series, orthogonal splines.

1. Problem and Method

The modified Fourier method was proposed and investigated earlier [1] in parabolic problems for regions with a noncanonical curvilinear boundaries. This method connected with explicit difference scheme is similar to another variant of Fourier method connected with implicit difference scheme and proposed here. Convergence of approximate analytic solutions was obtained for this method earlier only with respect to eigenvalues and functions in the framework of the Sturm-Liouville problem. Here is proposed full investigation of convergence of approximate solutions obtained in form of finite Fourier series for novel variant of Fourier method connected with orthogonal splines in parabolic initial boundary value problems. An estimate is obtained, which shows a high rate of convergence of such finite Fourier series to exact solutions of problems for regions with a noncanonical curvilinear boundaries.

The method of separation of variables (Fourier method) allows finding solutions in analytical form of many initial boundary value problems. The method is connected with the Sturm-Liouville problem and in many cases of initial boundary value problems with using of special functions. Implementation of classical Fourier method for many types of initial boundary value problems, including problems to which all parts of the boundary of a canonical region are coordinate lines or surfaces, meets with significant difficulties. One way to expand the scope of the classical Fourier method is to solve mathematical questions related to structure of boundary conditions. Special functions appear in the algorithm of the Fourier method when a Sturm-Liouville problem is solved in curvilinear coordinate systems in cases of canonical regions whose boundaries are coordinate lines or surfaces. In the general case of initial boundary value problems for noncanonical regions with curvilinear boundaries, the use of special functions is inefficient. The classical Fourier method is applicable only in initial boundary value problems for canonical regions of classical shape, in particular, in solving contact problems for elastic bodies. The applications of the classical Fourier method are given in many articles.

Other directions of development of different methods for solving initial boundary value problems for noncanonical regions with curvilinear boundaries are associated, first, with the application of finite element method and finite difference method, and, secondly, with a modification of the Fourier method itself. Finite difference methods and finite element methods have wide scopes. But numerical methods not give solutions in form of Fourier series, which are used in many applications. Scope of spline approximations also is enough wide. The generalized Fourier method associated with the use of orthogonal splines was proposed for parabolic initial boundary value problems in the article [1]. It gives solutions in form of finite Fourier series. This method, thanks to orthogonal splines, has expanded scope which contains initial boundary value problems for noncanonical regions with any curvilinear boundaries. Used in this article and here finite Fourier series, based on orthogonal splines, shows high efficiency also in problems of approximation of functions in regions with curvilinear boundaries and generates fast algorithm of approximations.

The parabolic initial boundary value problem

$$L[u] = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t} \quad \forall (x,y) \in S, \quad \forall t \geq 0,$$
$$u|_{t=0} = \varphi(x,y) \quad \forall (x,y) \in S; \quad u|_{\partial S} = 0 \quad \forall t \geq 0, \tag{1}$$

is considered. Here ∂S is a piecewise smooth curvilinear boundary of the noncanonical region S, $u = u(x,y,t)$ – a function, continuous $\forall t \geq 0$ in a closed region $\overline{S} = S + \partial S$, $a^2 = const > 0$.

According to the Fourier method, the solutions of the problem (1) is sought as a product of two functions $u(x,y,t) = U(x,y) \cdot V(t)$. Substitution of this product in (1) and separation of variables U, V leads to the equation with the parameter λ

$$\frac{dV}{dt} + \lambda V = 0 \quad \forall t \geq 0 \tag{2}$$

and to the Sturm-Liouville problem

$$L[U] + \lambda U = 0 \ (S), \quad U|_{\partial S} = 0. \tag{3}$$

The finite sum

$$U_N(x,y) = \sum_{i=0}^{N} \sum_{j=0}^{M} d_{ij} \gamma_i(x) \delta_j(y) \tag{4}$$

is used for approximation of U; $\gamma_i(x) = \varphi_i(x)$, $\delta_j(y) = \varphi_j(y)$ are orthogonal differentiable piecewise linear continuous splines on each specific grid. Here N, M are numbers of a grid nodes respectively for axes Ox, Oy, d_{ij} are unknown constant coefficients. The approximation (4) is a finite Fourier series for orthogonal splines. The Reissner variational principle is used to solve the Sturm-Liouville problem. It gives a system finite difference equations for coefficients d_{ij}. The eigenvalues of the Sturm-Liouville problem are excluding from this system of equations, as well as from the system of finite difference equations that are associated with equation (2). To do this, the orthogonality property of splines is used. The implicit difference scheme for node values of a solution $u(x, y, t)$ appears as the result. This algorithm of modified Fourier method is novel.

2. Convergence of Method

The convergence investigation of the proposed method uses here the theory of finite difference schemes. The approximate solutions of the parabolic initial boundary value problem (1) converge to an exact solution u of the problem (1) in a region $\overline{S}$, if $\Delta t = \alpha h, \alpha = const > 0$. It was investigated approximation of differential equations and stability of solutions of finite difference equations. The implicit finite difference equations together with a boundary condition and together with an initial condition approximate the problem (1) in a region S with an error of an order $O(h^2)$. The system of the finite difference equations is characterized by absolute stability. Convergence of solutions of finite difference equations is followed from it, if $\Delta t = \alpha h$, $\alpha = const > 0$. This means convergence

$$\left\| u - u^{(K)} \right\|_{W^0_{h,2}} \leq C_1 h^2$$

of values in grid nodes of a sequence of approximate solutions $u^{(K)}$ to an exact solution u when a grid step $h = \max(h_1, h_2)$ decreases. Here C_1 is some positive constant coefficient. Here, the Sobolev space $W^0_{h,2}$ on grid which is defined by the norm

$$\|u\|_{W^0_{h,2}} = \left(h^2 \sum_{i=0}^{N} \sum_{j=0}^{M} |u(x_i, y_j)|^2 \right)^{1/2}$$

associated with a grid in a region. From the convergence of numerical solutions of finite difference equations obtained at grid nodes at specified time points follows the convergence of a sequence of analytical approximate solutions in the form of finite Fourier series. The inequality

$$\left\| u - u^{(K)} \right\|_{W^0_2(S)} \leq C_2 h^2$$

defines this convergence. Here C_2 is some positive constant coefficient, $W^0_2(S)$ is the Sobolev space.

References

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