MSC2010 35S15

On an inverse boundary value problem with non-local integral terms condition for the pseudo-parabolic equation of the fourth order

A. T. Ramazanova [1], Y. T. Mehraliyev [2], S. I. Allahverdieva [3]
University Duisburg-Essen[1], Baku State University[2], Mingachevir State University[3]

Currently, problems with non-local conditions for partial differential equations are of great interest, which is due to the need to generalize the classical problems of mathematical physics in connection with the mathematical modeling of a number of physical processes studied by modern science [1]. Among non-local problems, of great interest are problems with integral conditions. Nonlocal integral conditions describe the behavior of the solution at interior points of the domain in the form of some mean. Examples include problems arising from the study of diffusion of particles in a turbulent plasma [1], the processes of heat propagation [2], [3] of the process of moisture transfer from capillary-simple media [4], as well as the study of some inverse problems of mathematical physics. In [5], a problem with nonlocal in time integral conditions for a hyperbolic equation was considered. In [6], an inverse boundary value problem with an unknown time-dependent coefficient was investigated for a fourth-order Boussinesq equation with nonlocal second-order integral in time. In the present paper, an inverse boundary value problem with nonlocal in time integral conditions for a fourth-order pseudoparabolic equation is investigated. Consider the equation

$$u_t(x,t) - bu_{txx}(x,t) + a(t)u_{xxxx}(x,t) = p(t)u(x,t) + f(x,t) \tag{1}$$

in the domain $D_T = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$ an inverse boundary problem with the non-local initial conditions

$$u(x,0) + \int\limits_0^T M(t)u(x,t)dt = \varphi(x)(0 \le x \le 1), \tag{2}$$

the boundary conditions

$$u_x(0,t) = 0, u(1,t) = 0, u_{xxx}(0,t) = 0, u_{xx}(1,t) = 0(0 \le t \le T), \tag{3}$$

and with the additional conditions

$$u(0,t) = h(t), (0 \le t \le T), \tag{4}$$

where $b > 0$ - given numbers, $a(t) > 0, f(x,t), \varphi(x), M(t) > 0, h(t)$ - given functions, $u(x,t)$ and $p(t)$ - desired functions. Denote

$$\tilde{C}^{4,1}(D_T) = \left\{u(x,t) : u(x,t) \in C^{2,1}(D_T), u_{txx}, u_{xxx}u_{xxxx} \in C(D_T)\right\}$$

Definition. The classical solution of the inverse boundary value problem (1)-(4) is the pair $\{u(x,t), p(t)\}$ of functions $u(x,t) \in \tilde{C}^{4,1}(D_T)$ and $p(t) \in C[0,T]$ satisfying equation (1) in D_T, condition (2) in $[0,1]$ and conditions (3)-(4) in $[0,T]$. The following theorem is true.

Theorem 1. Let $b > 0, 0 < a(t) \in C[0,T], \varphi(x) \in C[0,1], f(x,t) \in C(D_T), 0 \leq M(t) \in C[0,T]$, $h(t) \in C^1[0,T]$, $h(t) \neq 0$, $(0 \leq t \leq T)$, $\varphi(0) = h(0) + \int\limits_0^T M(t)h(t)dt$.

Then the problem of finding a solution to problem (1)-(4) is equivalent to the problem of determining the functions $u(x,t) \in \bar{C}^{4,1}(D_T)$ and $p(t) \in C[0,T]$, from (1)-(3) and

$$h'(t) - bu_{txx}(0,t) + a(t)u_{xxxx}(0,t) = p(t)h(t) + f(0,t)(0 \leq t \leq T) \tag{5}$$

After applying the formal scheme of the Fourier method, finding the first component $u(x,t)$ of any solution $\{u(x,t), p(t)\}$ to problem (1)-(3), (5) reduces to solving the following nonlinear integro-differential equation:

$$u(x,t) = \sum_{k-1}^{\infty} \left\{ \left(\varphi_k - \int_0^T M(t)u_k(t)dt \right) e^{-\int\limits_0^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds} + \right.$$

$$\left. + \frac{1}{1+b\lambda_k^2} \int_0^t F_k(\tau; u, p) e^{-\int\limits_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds} d\tau \right\} \cos \lambda_k x \tag{6}$$

where

$$\lambda_k = \frac{\pi}{2}(k-1)\, u_k(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx$$

$$\varphi_k = 2\int_0^1 \varphi(x)\cos\lambda_k x dx \,(k=1,2,...)\, F_k(t;u,p) = f_k(t) + p(t)u_k(t),$$

$$f_k(t) = 2\int_0^1 f(x,t)\cos\lambda_k x dx$$

Further, using equation (6), from condition (5) to determine the second component of any solution $U(p,t), p(t)$ of problem (1)-(3), (5) , we obtain the following nonlinear integral equation:

$$p(t) = [h(t)]^{-1} \left\{ h'(t) - f(0,t) + \sum_{k=1}^{\infty} \left[\frac{b\lambda_k^2}{1+b\lambda_k^2} F_k(t;u,p) + \frac{a(t)\lambda_k^4}{1+b\lambda_k^2} \times \right.\right.$$

$$\left.\left. \times \left(\left(\varphi_k - \int_0^T M(t)u_k(t)dt \right) e^{-\int\limits_0^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds} + \frac{1}{1+b\lambda_k^2} \int_0^t F_k(\tau;u,p) e^{-\int\limits_\tau^t \frac{a(s)\lambda_k^2}{1+b\lambda_k^2}ds} d\tau \right) \right] \right\}. \tag{7}$$

Thus, the solution of problem (1)-(3), (5) was reduced to the solution of system (6),(7), with respect to the unknown functions $U(x,t)$ and $p(t)$. Using the principle of compressed mappings, we prove the following theorem on the existence and uniqueness in a small solution of problem (1)-(3), (5) .

Theorem 2. Let conditions 1) - 3) be satisfied

1. $\varphi(x) \in W_2^{(5)}(0,1), \varphi'(0) = \varphi(1) = \varphi'''(0) = \varphi''(1) = \varphi^{(4)}(1) = 0;$
2. $f(x,t), f_x(x,t), f_{xx}(x,t) \in C(D_T), f_{xxx}(x,t) \in L_2(D_T), f_x(0,t) = f(1,t) = f_{xx}(1,t) = 0;$
3. $b > 0, 0 < a(t) \in C[0,T], 0 \leq M(t) \in C[0,T], h(t) \in C^1[0,T], h(t) \neq 0(0 \leq t \leq T).$

Then for sufficiently small values, problem (1)-(3), (5) has a unique solution.

Using Theorem 1, the last theorem immediately implies the unique solvability of the original problem (1)-(4).

Theorem 3. Let all the conditions of Theorem 2 and

$$\varphi(0) = h(0) + \int_0^T M(t)h(t)dt$$

be satisfied. Then, for sufficiently small values, problem (1)-(4) has a unique classical solution.

References

1. Samara A.A. On some problems of the theory of differential equations. Differential equa-tions 1980. Vol. 16, No. 11. pp. 1925-1935.

2. Cannon J.R. The solution of energy equation to the energy of. Quart. Appl. Math. 1963. Vol. 5, No. 21. pp. 1555-160.

3. Ionkin N.I. The solution of a boundary value problem of the theory of heat conduction with a nonclassical boundary condition. Differential Equations. 1977. Vol. 13, No. 2. pp. 294-304.

4. Nakhushev A.M. About one approximate method for solving boundary value problems for differential equations and its approximation to the dynamics of soil moisture and groundwater. Differential Equations. 1982. Vol. 18, No. 1. pp. 72-81.

5. Kirichenko S.V. On a boundary value problem with nonlocal in time conditions for a one-dimensional hyperbolic equation. Vestn. SamSU. Natural Science Ser. 2013. No. 6 (107). pp. 31–39.

6. Megraliev Ya.T., Alizade F.Kh. An inverse boundary value problem for a fourth-order Boussinesq equation with nonlocal integral conditions of the second kind. Vestn. Ud-murtsk. un-ty. Mat Fur. Computer Science. 2016. 26 (4). pp. 503–514.