MSC 45G05

Note on approximation of the exact solutions of weakly singular integral equations

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1. Introduction

It is well-known that the exact solutions of weakly singular integral equations may have unbounded derivatives near the bounds of domain (see, e.g. [2, 5]).

For a simplicity of presentation consider now the simplest case of one-dimensional Volterra integral equation of the form

$$x(t) - \int\limits_0^t \frac{h(t,\tau)}{(t-\tau)^\alpha} x(\tau) d\tau = f(t),\ t \in [0,T],\ 0 < \alpha < 1, \tag{1}$$

where the function $h(t,\tau)$ has bounded derivatives on each variable up to certain order r in square $[0,T]^2$, $f(t) \in C^{r,\alpha}(0,T]$. Then the exact solution $x(t) \in C^{r,\alpha}(0,T]$ ([2]). Here is a definition of this class:

Definition 1. *Let $0 \leqslant \alpha < 1$ and the function $f(t)$ has continuous derivatives up to order r for $t \in (0,T]$, satisfying the inequalities*

$$|f^{(k)}(t)| \leqslant \frac{A_k}{t^{k-1+\alpha}},\quad \textit{for all } t \in (0,T], k = 0,1,\dots,r.$$

Then $f(t) \in C^{r,\alpha}(0,T]$.

Since the exact solution of (1) belongs to $C^{r,\alpha}$, there is a necessity to investigate the optimal methods of approximation for this class.

2. Piecewise polynomial approximation

It is well-known (see, e.g. the papers [1, 2] by H. Brunner) that the polynomial spline approximation of functions from $C^{r,\alpha}$ in case of the uniform mesh $t_k = \frac{kT}{N},\ k = \overline{0,N}$, satisfies only $\|f - f_N\|_\infty = O(\frac{1}{N^{1-\alpha}})$. And this order is best possible for any $r \geqslant 1$. One can restore the optimal convergence rate $O(N^{-r})$ by using so-called graded meshes of the form

$$t_k = \left(\frac{k}{N}\right)^{\frac{r}{1-\alpha}} T,\ k = \overline{0,N}. \tag{2}$$

The mesh (2) is strongly nonuniform and there are some practical difficulties in its application in collocation methods for integral equations. In using graded meshes we have to start the collocation on a very small subinterval. This may create serious round-off errors in subsequent calculations, which will restrict applicability of the method (see, e.g. [3, 7]).

In this paper we suggest the new practical mesh which is less nonuniform than (2), but at the same time it gives us an equivalent approximation error for the functions from $C^{r,\alpha}(0,T]$.

The application of this mesh in practice (when the spline-collocation technique is used) allows to save $2-3$ orders of a theoretical accuracy. Such approach is also extended to the case of multidimensional integral equations at the construction of its numerical solutions [8].

References

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