XIII Международная научная конференция "Дифференциальные уравнения и их приложения в математическом моделировании", Саранск, 12-16 июля 2017.
XIII International scientific conference "Differential equations and their applications in mathematical modeling", Saransk, July 12-16, 2017.

MSC 37D15

On the growth of the number of non-compact heteroclinic curves*

V. Grines[1], E. Gurevich[1], O. Pochinka[1], A. Shilovskaya[2]

National Research University Higher School of Economics[1], Lobachevskii State University[2]

Abstract: We consider a class $SD(M^3)$ of gradient-like diffeomorphisms on closed 3-manifolds M^3 that have surface dynamics. In [1] it was proven that the ambient manifold M^3 for such diffeomorphisms is a mapping torus $M_{g,\tau}$, $g \geq 0$, and the number of non-compact heteroclinic curves is no less than $12g$. In this paper it is established that for any integer $n \geq 12g$ there exists a mapping torus $M_{g,\tau(n)}$ and a diffeomorphism from the class $SD(M_{g,\tau(n)})$ having exactly n heteroclinic curves.

Keywords: Heteroclinic curve, gradient-like diffeomorphism, mapping torus.

1. Introduction

S. Smale in his fundamental paper [2] introduced a class of dynamical systems on closed manifolds, called later Morse-Smale systems, and provided inequalities connecting a number of fixed points and periodic trajectories of such a system, and the topology of its ambient manifold. Conditions that determine Morse-Smale systems generalize the necessary and sufficient conditions for the roughness (structural stability) of flows on a two-dimensional sphere, obtained by A.A. Andronov and L.S. Pontryagin. Despite the fact that the Morse-Smale systems do not exhaust the class of structurally stable systems on manifolds of dimension two and higher (for cascades), and three and higher (for flows), they have been in the focus of attention of mathematicians for a long time. This is explained both by the importance of the Morse-Smale system for applications, and the remarkable interrelation between their dynamics and the topology of the carrying manifold. Resent research has shown that it is possible to get formulas connecting the number of periodic points with the topology of the carrying manifolds as well as an estimation of the number of heteroclinic curves. Such results can be applied for studying a topological structure of magnetic field (see, for example, [3]).

There are few fundamental results on the existence of heteroclinic curves for Morse-Smale 3-diffeomorphisms. In [4] the existence of heteroclinic curves was established for every Morse-Smale diffeomorphism given on a closed 3-manifold distinct from the 3-sphere S^3 and the connected sum of a finite number of copies of $S^2 \times S^1$. In [3] the existence of non-compact heteroclinic curves was proved for every polar 3-diffeomorphism (a diffeomorphism with a unique sink and a unique source) given on an irreducible 3-manifold (a manifold where each bi-collared 2-sphere bounds a 3-ball) what was effectively applied for the finding of heteroclinic separators of magnetic fields in electrically conducting fluids. By [5], if a polar diffeomorphism is given on a lens $L_{p,q}$ and its non-wandering set contains exactly two saddle points with trivially embedded one-dimensional manifolds then the wandering set contains at least p heteroclinic curves.

In [1] a class of diffeomorphisms with surface dynamics (SD-diffeomorphisms) were introduced and the exact lower estimate of the number of the non-compact heteroclinic curves for gradient-like SD-diffeomorphisms was given. We recall this result as Statements 1, 2 before formulating the main result of the present paper.

Everywhere below we will assume that f is an orientation preserving diffeomorphism given

*The publication was supported by the Russian Foundation for Basic Research (projects 15-01-03687-a, 16-51-10005-Ko_a) and the Basic Research Program at the HSE (project 90).

on an orientable closed smooth 3-manifold M^3.

We say that an Ω-stable diffeomorphism $f : M^3 \to M^3$ *has a surface dynamics (is SD-diffeomorphism)* if its non-wandering set Ω_f consists of two disjoint families Ω_+, Ω_- of basic sets such that the sets $\mathcal{A}_f = W^u_{\Omega_+}$ and $\mathcal{R}_f = W^s_{\Omega_-}$ are disjoint and every connected component of $\mathcal{A}_f$ and $\mathcal{R}_f$ is a locally flat orientable closed surface[1].

It was proven in [1] that sets $\mathcal{A}_f, \mathcal{R}_f$ consists of the same number k_f of connected components, all the components have the same genus g_f and the carring manifold M^3 is a mapping torus[2] M_{g_f,τ_f}. We will associate the numbers k_f, g_f with every SD-diffeomorphism f.

We will focus on gradient-like SD-diffeomorphisms. Let's recall that a diffeomorphism $f : M^n \to M^n$ of a connected closed smooth manifold M^n of the dimension n is called *a Morse-Smale diffeomorphism* if its non-wandering set Ω_f is finite and consists of the hyperbolic periodic points, and for different saddle periodic points $p, q \in \Omega_f$ the invariant manifolds W^s_p, W^u_q either are disjoint or intersect transversely. Let p, q are different saddle periodic points of a Morse-Smale diffeomorphism $f : M^n \to M^n$. If $dim(W^s_p \cap W^u_q) = 0$ then every point of the set $W^s_p \cap W^u_q$ is called a *heteroclinic point*. The diffeomorphism f is called a *gradient-like* if the condition $W^s_p \cap W^u_q \neq \emptyset$ leads to the fact $dim\ W^u_p < dim\ W^u_q$. So if the wandering set of f does not contain heteroclinic points, then f is a gradient-like.

Утверждение 1 ([1, Theorem 2]) *For any integer $g \geq 0$ and a diffeomorphism $\tau \colon S_g \to S_g$ there is a gradient-like SD-diffeomorphism on $M^3_{g,\tau}$.*

Let $f : M^3 \to M^3$ be a gradient-like diffeomorphism, p, q are its different saddle periodic points such that $dim(W^s_p \cap W^u_q) = 1$. Then every connected component of the set $W^s_p \cap W^u_q$ is called a *heteroclinic curve*.

Утверждение 2 ([1, Theorem 3])

1. *Let $f : M^3_{g_f,\tau_f} \to M^3_{g_f,\tau_f}$ be a gradient-like diffeomorphism with surface dynamics. Then a number of non-compact heteroclinic curves is not less than $12 g_f k_f$.*
2. *The estimation is exact, namely for every integers $k > 0, g \geq 0$ there is a gradient-like SD-diffeomorphism $f : S_g \times S^1 \to S_g \times S^1$ such that its wandering set contains exactly $12gk$ non-compact heteroclinic curves.*

The lower estimation of the number of the non-compact heteroclinic curves given in the Statement 2 in fact depends only on g and k and is reached on the direct product $S_g \times S^1$. We improve this estimation by the following.

Теорема 1 *For any integers $g \geq 1, k \geq 1$ and $n \geq 12gk$ there is a diffeomorphism $\tau(n)$: $S_g \to S_g$ and a gradient-like SD-diffeomoprhism on a mapping torus $M_{g,\tau(n)}$ whose number of non-compact heteroclinic curves equals n.*

Acknowledgements. The publication was supported by the Russian Foundation for Basic Research (project no. 15-01-03687-a) and the Basic Research Program at the HSE (project 90) in 2017.

[1] Let S_g be an orientable surface (closed 2-dimensional manifold) of a genus g and $e : S_g \to M^3$ be a topological embedding. A surface $S_g = e(S_g)$ is called *locally flat* if for every point $p \in S_g$ there exists a neighborhood $U_p \subset M^3$ and a homeomorphism $h_p : U_p \to R^3$ such that the set $h_p(S_g \cap U_p)$ is a coordinate plane in R^3. An orientable locally flat surface is a *bi-collared*, that is there exists a topological embedding $h : S_g \times [-1; 1] \to M^3$ such that $h(S_g \times \{0\}) = S_g$.

[2] A mapping torus $M^3_{g,\tau}$ is a factor space $S_g \times [0, 1]/\sim$, where $(z, 1) \sim (\tau(z), 0)$ for a diffeomoprhism $\tau : S_g \to S_g$ (*gluing map*) of the closed surface S_g of genus g.

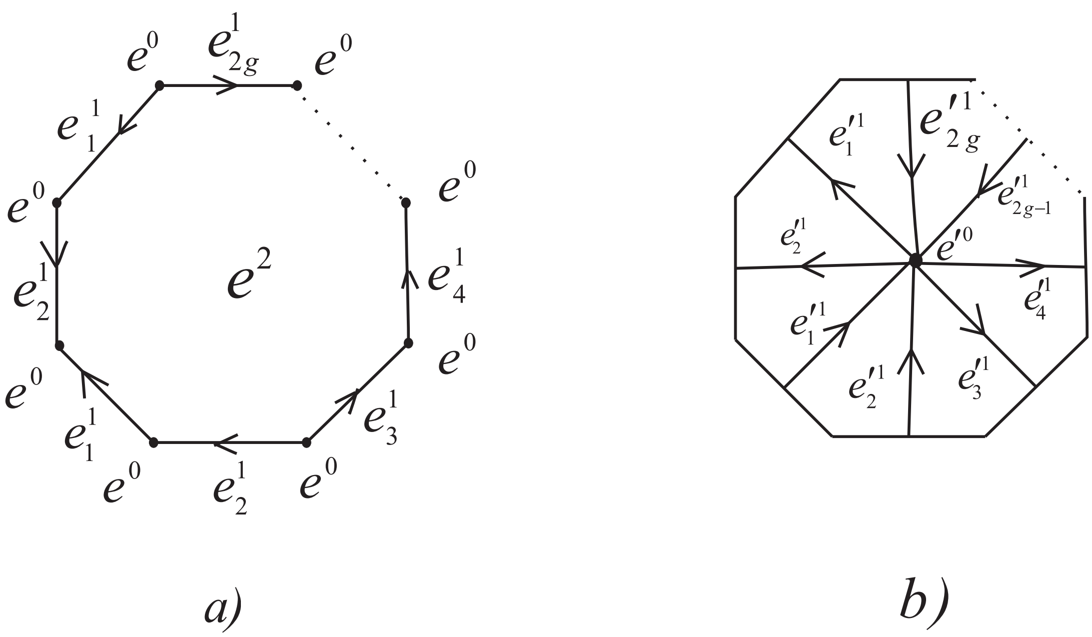

Рис. 1. Dual cell decompositions of the surface S_g

2. Construction of a SD-diffeomorphism with a given number of heteroclinic curves

2.1. Dehn twist

Let $c \in S_g$ be a smooth simple curve. A *Dehn twist along* c is a homeomorphism $\rho_c : S_g \to S_g$ defined in the following way. Consider a circle S^1 as a subset of the complex plane $\mathbf{C}$. Let $h : S^1 \times [-1,1] \to S_g$ be a diffeomorphism such that $h(S^1 \times \{0\}) = c$, and $g : S^1 \times [-1,1] \to S^1 \times [-1,1]$ a homeomorphism such that $g(z,r) = (z,r)$ for $z \in S^1, r \in [-1,0]$ and $g(z,r) = (ze^{2\pi ri}, r)$ for $r \in [0,1]$. Then

$$\rho_c(p) = \begin{cases} p, p \in S_g \setminus h(S^1 \times [-1,1]); \\ h(g(h^{-1}(p))), p \in h(S^1 \times [-1,1]). \end{cases}$$

For two closed smooth curves $c, c' \in S_g$ that intersect transversely denote by $N(c,c')$ a number of points in the set $c \cap c'$.

Let K, K' be dual cellular decompositions of S_g such that the 1-dimensional cells of K are arcs $e_1^1, ..., e_{2g}^1$, and the 1-dimensional cells of K' are arcs ${e'}_1^1, ..., {e'}_{2g}$ represented in the figure 2.1, a),b) correspondently. Notice that the both families represent generators of the group $H_1(S_g, Z)$.

Dehn twists $\rho_{e_1^1}$ along the curve e_1^1 keeps homology classes $[e_j^1]$ for $j \in \{3, ..., 2g\}$ and acts on classes $[e_1^1], [e_2^1]$ in the following way: $\rho_{e_1^1}([e_1^1]) = [e_1^1]$, $\rho_{e_1^1}([e_2^1]) = [e_1^1 \pm e_2^1]$. This observation immediately leads to the following lemm.

Лемма 1

1. $N(e_1^1, \rho_{e_1^1}^m({e'}_2^1)) = 0$;

2. $N(e_1^1, \rho_{e_1^1}^m({e'}_1^1)) = 1$;

3. $N(e_2^1, \rho_{e_1^1}^m({e'}_2^1)) = 1$;

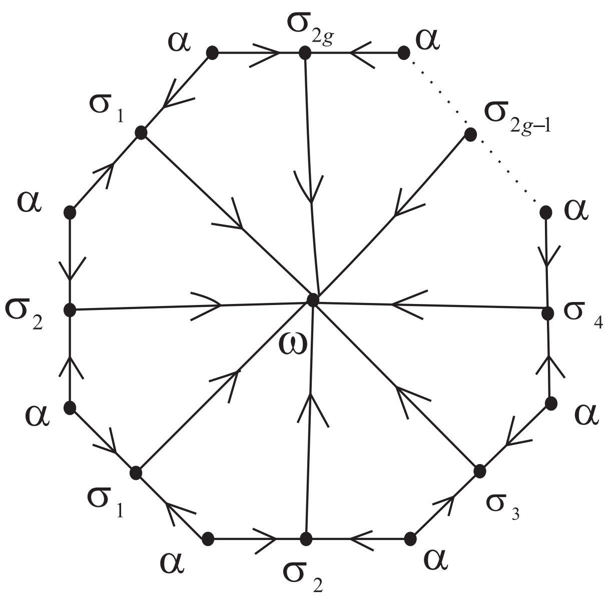

Рис. 2. Morse-Smale diffeomorphism on a surface of a genus g

4. $N(e^1_2, \rho^m_{e^1_1}(e'^1_1)) = m$;

5. $N(e^1_i, \rho^m_{e^1_1}(e^1_j)) = 1$ *if* $i = j$ *and* 0 *otherwise,* $i, j \in \{3, 4, \ldots, 2g\}$.

2.2. Proof of the Theorem 1

To proof the Theorem 1 we use a scheme of building a gradient-like SD-diffeomorphism for given g, k and τ, that was suggested in [1], but clarify the diffeomorphism τ to get the desire number n of heteroclinic curves.

Let $g, k, n \geq 12gk$ are arbitrary integers.

Denote by $\psi : [0; 1] \to [0, 1]$ a time-1 map of the flow $\dot{r} = \sin 2\pi kr$, and by $\varphi^t_g : S_g \to S_g$ a gradient-like flow whose non-wandering set consists of exactly one sink ω, one source α and $2g$ saddle equilibria $\sigma_1, \ldots, \sigma_{2g}$. Figure 2.2 shows an unfolding of the surface S_g as a $2g$-gon and a phase portrait of the flow φ^t_g on it. Denote by f_0 the time-1 map of the flow φ^t_g. Put $e^1_i = W^s_{\sigma_i}$, $e'^1_i = W^u_{\sigma_i}$, $i \in \{1, ..., 2g\}$. Set of arcs $\Gamma^s = \{e^1_1, ..., e^1_{2g}\}$ and $\Gamma^u = \{e'^1_1, ..., e'^1_{2g}\}$ form 1-dimensional cells of the dual decompositions K, K'.

Let $m = n - 12gk$, $\tau_m = \rho^m_{e^1_1}$, where $\rho_{e^1_1} : S_g \to S_g$ is the Dehn twist along e^1_1. Put $f_1 = \tau^{-1}_m f_0 \tau_m$, and, finally, $\varphi^{[t]}_g$ is a time-t map along trajectories of the φ^t_g.

Remark that

(*) Γ^u is transversal to $\tau_m(\Gamma^s)$;

(**) $\tau_m(\alpha) \notin (\Gamma^u \cup \omega)$ and $\omega \notin \tau_m(\Gamma^s \cup \alpha)$.

Choose $r_0 \in (1 - \frac{1}{2k}, 1)$, put $r_1 = \psi^{-1}(r_0)$, $r_2 = \psi^{-1}(r_1)$ $(r_0 < r_1 < r_2)$ and define a diffeomorphism $F : S_g \times [0; 1] \to S_g \times [0; 1]$ by the formula

$$F(z,r)=\begin{cases}(f_0(z),\psi(r)), r\in[0;r_0];\\(\varphi_g^{[\frac{r_1-r}{r_1-r_0}]}(z),\psi(r)), r\in[r_0;r_1];\\(\tau_m^{-1}\varphi_g^{[\frac{r-r_1}{r_2-r_1}]}\tau_m(z),\psi(r)), r\in[r_1;r_2];\\(f_1(z),\psi(r)), r\in[r_2;1].\end{cases}$$

Denote by $\pi_{\tau_m}: S_g\times[0,1]\to M_{g,\tau}$ the natural projecture and by $\widetilde{F}: M_{g,\tau}\to M_{g,\tau}$ a diffeomorphism such that $\widetilde{F}=\pi_{\tau_m}F\pi^{-1}_{\tau_m}$. By construction, the non-wandering set of diffeomorphism $\widetilde{F}$ is finite, hyperbolic and belongs to surfaces $\pi_{\tau_m}((S_g\times\{\frac{i}{2k}\})$, $i\in\{0,\dots,k\}$. The wandering set of F contains:

- exactly $8gk$ non-compact heteroclinic curves belonging to the union $\pi_\tau(S_g\times\{\frac{i}{2k}\})$, $i\in\{1,2,\dots,k\}$;
- exactly $4gk-2g$ non-compact heteroclinic curves belonging to the set $\pi_\tau((S_g\times[0,1-\frac{1}{2k}])\setminus(S_g\times\{\frac{i}{2k}\}))$.

To complete the proof of the theorem we are going to prove that the diffeomorphism $\widetilde{F}$ is a gradient like and its non-wandering set contains exactly n non-compact heteroclinic curves. It is enough to show that in the region $S_g\times(1-\frac{1}{2k},1)$ one-dimensional saddle separatrices of diffeomorphism F do not intersect any other saddle separatrices and two-dimensional manifolds of saddle points of F have a transversal intersection consisting exactly of $m+2g$ connected components.

For this aim notice that a region $D=S_g\times[r_1;r_2]$ is a fundamental domain of the restriction $F|_{S_g\times(1-\frac{1}{2k},1)}$. It follows from the construction of the diffeomorphism F that the two-dimensional stable separatrices intersect D along $\Gamma^s\times[r_1;r_2]$, two-dimensional unstable separatrices intersect D along $\tau^{-1}(\Gamma^u)\times[r_1;r_2]$, one-dimensional stable separatrices intersect D along $\alpha\times[r_1;r_2]$ and one-dimensional unstable separatrices intersect D along $\tau^{-1}(\omega)\times[r_1;r_2]$. Due to (*) two-dimensional manifolds of saddle points of F have a transversal intersection in D and, hence, in $S_g\times(1-\frac{1}{2k},1)$. Due to corollary 1 the number of connected components of this intersection is $2g+m$. Due to (**) one-dimensional saddle separatrices do not intersect any other saddle separatrices in D and, hence, in $S_g\times(1-\frac{1}{2k},1)$.

References

1. Grines V., Gurevich E., Pochinka O. On the number of non-compact heteroclinic curves of diffeomorphisms with a surface dynamics. // Regular and Chaotic Dynamics. 2017. V. 22. N. 2. P. 122-135.
2. Smale S. Morse inequalities for a dynamical system. Bull. Amer. Math. Soc. 66 (1960), no. 1, 43–49.
3. Grines V., Medvedev T., Pochinka O., Zhuzhoma E. On heteroclinic separators of magnetic fields in electrically conducting fluids.// Physica D: Nonlinear Phenomena. 2015. V. 294. P. 1-5.
4. Bonatti C. , Grines V., Medvedev V., Pecou E. Three-manifolds admitting Morse-Smale diffeomorphisms without heteroclinic curves*Topology and its Applications*, 2002, vol. 117, pp. 335 – 344.
5. Grines V., Zhuzhoma E. V., Medvedev T. V. New relations for Morse-Smale systems with trivially embedded one-dimensional separatrices.// Sbornik: Mathematics. 2003. V. 194. N. 7. P. 979—1007.