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A time nonlocal inverse boundary-value problem for a second-order hyperbolic equation with integral conditions

E. I. Azizbayov[1], Y. T. Mehraliyev[2]

[1]Department of Computational Mathematics, Baku State University,
[2]Department of Differential and Integral equations, Baku State University.
e-mail: eazizbayov@bsu.edu.az

Abstract: This paper studies a time nonlocal inverse boundary-value problem for a second-order hyperbolic equation. First, we introduce a definition of a classical solution, and then the original problem is reduced to an equivalent problem. Further, the existence and uniqueness of the solution of the equivalent problem is proved using a contraction mapping. Finally, using the equivalency, the existence and uniqueness of classical solution is obtained.

Keywords: Inverse value problem; hyperbolic equation; nonlocal integral condition; classical solution.

1. Introduction

Recently, problems with nonlocal conditions for partial differential equations have been of great interest, which is caused by the need to generalize the classical problems of mathematical physics in connection with the mathematical modeling of a number of physical processes that are studied by modern natural science [1]. Note that most of the publications about problems with spatially nonlocal conditions and integral conditions for partial differential equations are found in [2]– [6]. In [6], a problem of time nonlocal integral conditions for hyperbolic conditions is investigated.

There are many cases when the requirements of practice lead to the problems of determining the coefficients or the right-hand side of the differential equation from some known data from its solution. Such problems were called inverse problems of mathematical physics. Inverse problems represent an actively developing branch of contemporary mathematics.

In this article we study a time nonlocal inverse boundary-value problem for second-order hyperbolic equation with integral conditions.

2. Formulation of the problem

Let $T > 0$ be a fixed number and denote by $D_T := \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$. We consider a nonlocal inverse boundary value problem for a hyperbolic equation

$$u_{tt}(x,t) - u_{xx}(x,t) = a(t)u(x,t) + f(x,t) \tag{1}$$

in the rectangle domain D_T, with initial conditions of integral form

$$u(x,0) + \int_0^T M_1(x,t)u(x,t)dt = \varphi(x) \quad (0 \le x \le 1), \tag{2}$$

$$u_t(x,0) + \int_0^T M_2(x,t)u(x,t)dt = \psi(x) \quad (0 \le x \le 1), \tag{3}$$

subject to boundary conditions

$$u(0,t) = u_x(1,t) = 0 \quad (0 \le t \le T), \tag{4}$$

as well as additional condition

$$u(1,t) = h(t) \quad (0 \le t \le T), \tag{5}$$

where $f(x,t), \varphi(x), \psi(x), M_1(x,t), M_2(x,t), h(t)$ are given functions, $u(x,t)$ and $a(t)$ are the sought functions.

Definition. By a classical solution of the inverse boundary value problem (1)-(5) we understand a pair of functions $\{u(x,t), a(t)\}$ such that $u(x,t) \in C^2(D_T)$, $a(t) \in C[0,T]$ and relations (1)-(5) hold.

To investigate problem (1) - (5) we consider the auxiliary problem. It is required to determine a pair of functions $\{u(x,t), a(t)\}$ such that and $u(x,t) \in C^2(D_T)$, $a(t) \in C[0,T]$ from relations (1)-(4) and

$$h''(t) - u_{xx}(1,t) = a(t)h(t) + f(1,t) \quad (0 \le t \le T). \tag{6}$$

Analogously [7], proved the following

Lemma 1. Assume the following conditions are satisfied: $\varphi(x), \psi(x) \in C[0,1]$, $h(t) \in C^2[0,T]$, $h(t) \ne 0 \quad (0 \le t \le T)$, $f(x,t), M_1(x,t), M_2(x,t) \in C(D_T)$, and the compatibility conditions

$$h(0) + \int_0^T M_1(1,t)h(t)dt = \varphi(1),$$

$$h'(0) + \int_0^T M_2(1,t)h(t)dt = \psi(1).$$

Then the problem of finding a classical solution of (1)-(5) is equivalent to the problem of determining functions $u(x,t) \in C^2(D_T)$ and $a(t) \in C[0,T]$ from (1)-(4), (6).

3. Solvability of inverse boundary-value problem

We shall seek the first component $u(x,t)$ of classical solution $\{u(x,t), a(t)\}$, of the problem (1)-(4), (6) in the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin \lambda_k x \quad \left(\lambda_k = \frac{\pi}{2}(2k-1)\right), \tag{7}$$

where

$$u_k(t) = 2\int_0^1 u(x,t) \sin \lambda_k x dx \quad (k = 1, 2, \ldots)$$

are twice differentiable functions on the interval $[0,T]$. Then applying the formal scheme of the Fourier method, from (1) and (2) we have

$$u_k''(t) + \lambda_k^2 u_k(t) = F_k(t; u, a) \quad (k = 1, 2 \ldots; 0 \le t \le T), \tag{8}$$

$$u_k(0) = \varphi_k - M_{1k}(u), \quad u_k'(0) = \psi_k + M_{2k}(u) \quad (k = 1, 2 \ldots). \tag{9}$$

where

$F_k(t;u,a)=f_k(t)+a(t)u_k(t), \quad f_k(t)=2\int\limits_0^1 f(x,t)\sin\lambda_k x dx,$

$\varphi_k=2\int\limits_0^1 \varphi(x)\sin\lambda_k x dx, \quad \psi_k=2\int\limits_0^1 \psi(x)\sin\lambda_k x dx,$

$M_{1k}(u)=2\int\limits_0^1\left(\int\limits_0^T M_1(x,t)u(x,t)dt\right)\sin\lambda_k x dx,$

$M_{2k}(u)=2\int\limits_0^1\left(\int\limits_0^T M_2(x,t)u(x,t)dt\right)\sin\lambda_k x dx, \quad (k=1,2,\ldots).$

Solving problem (8)-(9) we obtain

$u_k(t)=(\varphi_k-M_{1k}(u))\cos\lambda_k t+\frac{1}{\lambda_k}(\psi_k-M_{2k}(u))\sin\lambda_k t$

$$+\frac{1}{\lambda_k}\int\limits_0^t F_k(\tau;u,a)\sin\lambda_k(t-\tau)d\tau \quad (k=1,2,\ldots). \tag{10}$$

To determine the first component $u(x,t)$ of classical solution of the problem (1)-(4), (6), by virtue (10), from (7) we get

$u(x,t)=\sum\limits_{k=1}^{\infty}\Big\{(\varphi_k-M_{1k}(u))\cos\lambda_k t+\frac{1}{\lambda_k}(\psi_k-M_{2k}(u))\sin\lambda_k t$

$$+\frac{1}{\lambda_k}\int\limits_0^t F_k(\tau;u,a)\sin\lambda_k(t-\tau)d\tau\Bigg\}\sin\lambda_k x. \tag{11}$$

By virtue of (7), it follows from (6) that

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f(1,t)+\sum_{k=1}^{\infty}\lambda_k^2(-1)^{k+1}u_k(t)\right\}. \tag{12}$$

To determine the second component $a(t)$ of classical solution of the problem (1)-(4), (6), taking into account (10), in (7) we obtain

$a(t)=[h(t)]^{-1}\Big\{h''(t)-f(1,t)+\sum\limits_{k=1}^{\infty}\lambda_k^2(-1)^{k+1}\left[(\varphi_k-M_{1k}(u))\cos\lambda_k t\right.$

$$\left.+\frac{1}{\lambda_k}(\psi_k-M_{2k}(u))\sin\lambda_k t+\frac{1}{\lambda_k}\int\limits_0^t F_k(\tau;u,a)\sin\lambda_k(t-\tau)d\tau\right]\Bigg\}. \tag{13}$$

Thus, the solution of problem (1) - (4), (6) was reduced to the solution of the system (11), (13) with respect to the unknown functions $u(x,t)$ and $a(t)$.

To study the uniqueness of the solution of problem (1) - (4), (6), the following assertion plays an important role.

Lemma 2. If $\{u(x,t),a(t)\}$ is a classical solution of (1)-(4), (6) then the functions

$$u_k(t)=2\int\limits_0^1 u(x,t)\sin\lambda_k x dx \quad (k=1,2,\ldots)$$

satisfy counting system (10), on the interval $[0,T]$.

Proof. Suppose that $\{u(x,t),a(t)\}$ is a classical solution of problem (1)-(4), (6). By multiplying both sides of equation (1) by the functions $2\sin\lambda_k x \quad (k=1,2,\ldots)$, then integrating obtained equality with respect to x from 0 to 1 and using the following relations

$$2\int\limits_0^1 u_{tt}(x,t)\sin\lambda_k x dx = \frac{d^2}{dt^2}\left(2\int\limits_0^1 u(x,t)\sin\lambda_k x dx\right) = u_k''(t) \quad (k=1,2,\ldots),$$

$$2\int\limits_0^1 u_{xx}(x,t)\sin\lambda_k x dx = -\lambda^2\left(2\int\limits_0^1 u(x,t)\sin\lambda_k x dx\right) = -\lambda_k^2 u_k(t) \quad (k=1,2,\ldots),$$

we conclude that condition (8) is satisfied.

Analogously, from (2) and (3) we obtain that conditions (9) holds true.

Thus $u_k(t)$ $(k=1,2,...)$ are solutions of problems (8), (9). And from this, it directly follows that the functions $u_k(t)$ $(k=1,2,...)$ satisfy system (8) on the interval $[0,T]$. The lemma is thus proved.

It's obvious that if $u_k(t) = 2\int\limits_0^1 u(x,t)\sin\lambda_k x dx$ $(k=1,2,...)$ are solutions of system (10), then a functions $u(x,t) = \sum\limits_{k=1}^{\infty} u_k(t)\sin\lambda_k x$ and $a(t)$ are also solutions of system (11), (13).

From Lemma 2 it follows that

Corollary. Suppose that systems (11), (13) have a unique solution. Then the problem (1) - (4), (6), couldn't have more than one solution, in other words, if problem (1)-(4), (6) have a solution, then it is unique.

With the purpose to study problem (1)-(4), (6) consider the following spaces:

Let $B_{2,T}^3$ [8] denote the set of all functions of the form

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\sin\lambda_k x \quad \left(\lambda_k = \frac{\pi}{2}(2k-1)\right),$$

considered in domain D_T, where each function from $u_k(t)$ $(k=1,2,...)$, is continuous on $[0,T]$ and satisfy the following condition

$$J(u) \equiv \left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} < +\infty.$$

The norm in this space is defined as follows

$$\|u(x,t)\|_{B_{2,T}^3} = J(u).$$

We denote by E_T^3, the Banach space $B_{2,T}^3 \times C[0,T]$ of vector functions $z(x,t) = \{u(x,t), a(t)\}$ with norm

$$\|z(x,t)\|_{E_T^3} = \|u(x,t)\|_{B_{2,T}^3} + \|a(t)\|_{C[0,T]}.$$

It is known that $B_{2,T}^3$ and E_T^3 are Banach spaces.

Now consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\}$$

in the space E_T^3 where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_k(t)\sin\lambda_k x,$$

$$\Phi_2(u,a) = \tilde{a}(t)$$

and the functions $\tilde{u}_k(t)$ $(k=1,2,...)$, $\tilde{a}(t)$ are equal to the right-hand sides of (10) and (13) respectively.

It is easy to see that

$$\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|\tilde{u}_k(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} \leq \sqrt{6}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_k\right|\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{6}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|M_{1k}(u)\right|\right)^2\right)^{\frac{1}{2}}+\sqrt{6}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\psi_k\right|\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{6}\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|M_{2k}(u)\right|\right)^2\right)^{\frac{1}{2}}+\sqrt{6T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_k(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$+\sqrt{6}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}. \tag{14}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]}$$

$$\times\left\{\left\|h''(t)-f(1,t)\right)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_k\right|\right)^2\right)^{\frac{1}{2}}\right.\right.$$

$$+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|M_{1k}(u)\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\psi_k\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^2\left|M_{2k}(u)\right|\right)^2\right)^{\frac{1}{2}}$$

$$\left.\left.+\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_k(\tau)\right|\right)^2 d\tau\right)^{\frac{1}{2}}+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\right]\right\}. \tag{15}$$

Assume that the data for problem (1)-(4), (6) satisfy the following conditions

1) $\varphi(x) \in C^2[0,1]$, $\varphi'''(x) \in L_2(0,1)$ and $\varphi(0)=\varphi'(1)=\varphi''(0)=0$;
2) $\psi(x) \in C^1[0,1], \psi''(x) \in L_2(0,1)$ and $\psi(0)=0, \psi'(1)=0$;
3) $f(x,t), f_x(x,t) \in C(D_T), f_{xx}(x,t) \in L_2(D_T),\ f(0,t)=f_x(1,t)=0 \ \ (0 \leq t \leq T)$;
4) $M_1(x,t), M_{1x}(x,t), M_{1xx}(x,t), M_{1xxx}(x,t) \in C(D_T), M_{1x}(0,t)=M_{1x}(1,t)=0$ $(0 \leq t \leq T)$;
5) $M_2(x,t), M_{2x}(x,t), M_{2xx}(x,t) \in C(D_T), M_2(1,t)=0 \ \ (0 \leq t \leq T)$;
6) $h(t) \in C^2[0,T], h(t) \neq 0 \ \ (0 \leq t \leq T)$.

Suppose that $M_{1x}(0,t)=M_{1x}(1,t)=0 \ \ (0 \leq t \leq T)$. By using

$$u(0,t)=u_x(1,t)=u_{xx}(1,t)=0 \ \ (0 \leq t \leq T)$$

we have

$$\int_0^T M_1(0,t)u(0,t)dt=0,$$

$$\frac{d}{dx}\left(\int_0^T M_1(x,t)u(x,t)dt\right)\Bigg|_{x=1}=\int_0^T (M_{1x}(1,t)u(1,t)+M_1(1,t)u_x(1,t))dt=0;$$

$$\frac{d^2}{dx^2}\left(\int_0^T M_1(x,t)u(x,t)dt\right)\Bigg|_{x=0}$$

$$=\int_0^T (M_{1xx}(0,t)u(0,t)+2M_{1x}(0,t)u_x(0,t)+M_1(0,t)u_{xx}(0,t))dt=0;$$

$$\frac{d^3}{dx^3}\left(\int\limits_0^T M_1(x,t)u(x,t)dt\right)\Bigg|_{x=0} = \int\limits_0^T (M_{1xxx}(x,t)u(x,t) + 3M_{1xx}(x,t)u_x(x,t)$$
$$+3M_{1x}(x,t)u_{xx}(x,t) + M_1(x,t)u_{xxx}(0,t))dt \in L_2(0,1).$$

Obviously,

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|M_{1k}(u)\right|\right)^2\right)^{\frac{1}{2}} \le \left\|\frac{d^3}{dx^3}\left(\int\limits_0^T M_1(x,t)u(x,t)dt\right)\right\|_{L_2(0,1)}$$
$$= \left\|\int\limits_0^T (M_{1xxx}(x,t)u(x,t) + 3M_{1xx}(x,t)u_x(x,t) + 3M_{1x}(x,t)u_{xx}(x,t) + M_1(x,t)u_{xxx}(x,t))dt\right\|_{L_2(0,1)}$$
$$= \|M_{1xxx}(x,t)\|_{C(D_T)}\left\|\int\limits_0^T u(x,t)dt\right\|_{L_2(0,1)} + 3\,\|M_{1xx}(x,t)\|_{C(D_T)}\left\|\int\limits_0^T u_x(x,t)dt\right\|_{L_2(0,1)}$$
$$+3\,\|M_{1x}(x,t)\|_{C(D_T)}\left\|\int\limits_0^T u_{xx}(x,t)dt\right\|_{L_2(0,1)} + \|M_1(x,t)\|_{C(D_T)}\left\|\int\limits_0^T u_{xxx}(x,t)dt\right\|_{L_2(0,1)}. \quad (16)$$

On the other hand, it's clear that

$$\left|\int\limits_0^T u_{xxx}(x,t)dt\right| \le \sum_{k=1}^{\infty}\lambda_k^3\left|\int\limits_0^T u_k(t)dt\right|. \quad (17)$$

Hence we find

$$\int\limits_0^1\left(\int\limits_0^T u_{xxx}(x,t)dt\right)^2 dx \le \left\|\int\limits_0^T u_{xxx}(x,t)dt\right\|_{L_2(0,1)}^2$$
$$\le \frac{T}{2}\sum_{k=1}^{\infty}\lambda_k^3\,\|u_k(t)\|_{C[0,T]}\left(2\int\limits_0^1\left(\int\limits_0^T u_{xxx}(x,t)dt\right)\sin\lambda_k x dx\right)$$
$$\le \frac{T}{2}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\left(2\int\limits_0^1\left(\int\limits_0^T u_{xxx}(x,t)dt\right)\sin\lambda_k x dx\right)^2\right)^{\frac{1}{2}}$$
$$\le \frac{T}{2}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\left\|\int\limits_0^T u_{xxx}(x,t)dt\right\|_{L_2(0,1)},$$

or

$$\left\|\int\limits_0^T u_{xxx}(x,t)dt\right\|_{L_2(0,1)} \le \frac{T}{2}\,\|u(x,t)\|_{B_{2,T}^3}. \quad (18)$$

Analogously,we can prove that

$$\left\|\int\limits_0^T u(x,t)dt\right\|_{L_2(0,1)} \le \frac{T}{2}\,\|u(x,t)\|_{B_{2,T}^3},$$

$$\left\| \int_0^T u_x(x,t)dt \right\|_{L_2(0,1)} \leq \frac{T}{2} \|u(x,t)\|_{B^3_{2,T}},$$

$$\left\| \int_0^T u_x(x,t)dt \right\|_{L_2(0,1)} \leq \frac{T}{2} \|u(x,t)\|_{B^3_{2,T}}, \tag{19}$$

From (16), by (18) and (19), we conclude

$$\left(\sum_{k=1}^{\infty} \left(\lambda_k^3 |M_{1k}(u)| \right)^2 \right)^{\frac{1}{2}} \leq (\|M_{1xxx}(x,t)\|_{C(D_T)} + 3\|M_{1xx}(x,t)\|_{C(D_T)}$$

$$+3\|M_{1x}(x,t)\|_{C(D_T)} + \|M_1(x,t)\|_{C(D_T)})\frac{T}{2}\|u(x,t)\|_{B^3_{2,T}}. \tag{20}$$

Similarly to the way it was done in obtaining estimate (20), we have

$$\left(\sum_{k=1}^{\infty} \left(\lambda_k^2 |M_{2k}(u)| \right)^2 \right)^{\frac{1}{2}} \leq (\|M_{2xx}(x,t)\|_{C(D_T)}$$

$$+\|M_{2x}(x,t)\|_{C(D_T)} + \|M_2(x,t)\|_{C(D_T)})\frac{T}{2}\|u(x,t)\|_{B^3_{2,T}}. \tag{21}$$

Then from (14) and (15), taking into account (20) and (21), we find

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \leq A_1(T) + B_1(T)\|u(x,t)\|_{B^3_{2,T}}(\|a(t)\|_{C[0,T]} + 1), \tag{22}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T)\|u(x,t)\|_{B^3_{2,T}}(\|a(t)\|_{C[0,T]} + 1), \tag{23}$$

where

$A_1(T) = \sqrt{6}\|\varphi'''(x)\|_{L_2(0,1)} + \sqrt{6}\|\psi''(x)\|_{L_2(0,1)} + \sqrt{6T}\|f_{xx}(x,t)\|_{L_2(D_T)}$,

$B_1(T) = \frac{\sqrt{6}}{2}(\|M_{1xxx}(x,t)\|_{C[0,T]} + 3\|M_{1xx}(x,t)\|_{C(D_T)} + 3\|M_{1x}(x,t)\|_{C[0,T]} + 2)T$,

$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\{\|h''(t) - f(1,t)\|_{C[0,T]}$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\varphi'''(x)\right\|_{L_2(0,1)} + \left\|\psi''(x)\right\|_{L_2(0,1)} + \sqrt{T}\|f_{xx}(x,t)\|_{L_2(D_T)}\right]\Bigg\},$$

$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^{-2})\right)^{\frac{1}{2}}$

$\times[\|M_{1xxx}(x,t)\|_{C[0,T]} + \|M_{1xx}(x,t)\|_{C(D_T)} + \|M_{1x}(x,t)\|_{C(D_T)}$

$+\|M_1(x,t)\|_{C(D_T)} + \|M_{2xx}(x,t)\|_{C(D_T)}$

$+\|M_{2x}(x,t)\|_{C(D_T)} + \|M_2(x,t)\|_{C(D_T)} + 2]\frac{T}{2}$.

Further, from the estimates (22) and (23) it follows that

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \leq A(T) + B(T)\|u(x,t)\|_{B^3_{2,T}}(\|a(t)\|_{C[0,T]} + 1) \tag{24}$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T).$$

Theorem 1. If conditions 1) - 6) and the condition

$$B(T)(A(T) + 3) < 1, \tag{25}$$

hold, then problem (1)-(4), (6) has a unique solution in the ball $K = K_R(\|z\|_{E^3_T} \leq A(T) + 2)$ of the space E^3_T.

Proof. In the space E^3_T we consider the equation

$$z = \Phi z, \tag{26}$$

where $z = \{u, a\}$, $\Phi = \{\Phi_1(u, a), \Phi_2(u, a)\}$, and the components $\Phi_i(u, a)$ $(i = 1, 2)$, of operator $\Phi(u, a)$ defined by the right side of equations (11) and (13).

Consider the operator $\Phi(u, a)$, in the ball $K = K_R$ of the space E^3_T. Let's show that the operator Φ mapping the elements of ball $K = K_R$ into itself.

Similarly, with the aid of (24) we obtain that for any $z \in K_R$ the following inequality hold
$\|\Phi z\|_{E^3_T} \leq A(T) + B(T) \|u(x,t)\|_{B^3_{2,T}} (\|a(t)\|_{C[0,T]} + 1)$
$\leq A(T) + B(T)(A(T) + 2)(A(T) + 3)$.

Hence, taking (25) into account, the operator Φ acts in the ball.

Now, we show that the operator Φ is a contraction.

Indeed, for any $z_1, z_2 \in K_R$

$$\|\Phi z_1 - \Phi z_2\|_{E^3_T} \leq 2B(T)(A(T) + 2)(\|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}} + \|a_1(t) - a_2(t)\|_{C[0,T]})$$

is satisfied.

Then by (20), it is clear that the operator Φ on the set $K = K_R$ satisfy the conditions of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{z\} = \{u, a\}$, in the ball $K = K_R$, which is a solution of equation (26); i.e. in the sphere $K = K_R$ is the unique solution of the systems (11), (13). Then the function $u(x,t)$ as an element of space $B^3_{2,T}$, is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

Hence we conclude that the function $u_t(x,t)$ is continuous in the domain D_T.

Further, it is possible to verify that equation (1) and conditions (2), (3), (4), (6) are satisfied in the usual sense. Consequently, $\{u(x,t), a(t)\}$ is a solution of (1) - (4), (6), and by Lemma 2 it is unique in the ball $K = K_R$. The proof is complete.

From Theorem 1 and Lemma 2, it follows directly the following assertion.

Theorem 2. Suppose that all assumptions of Theorem 1, and the compatibility conditions

$$h(0) + \int_0^T M_1(1,t)h(t)dt = \varphi(1),$$

$$h'(0) + \int_0^T M_2(1,t)h(t)dt = \psi(1)$$

hold. Then problem (1) - (5) has a unique classical solution in the ball $K = K_R$ of the space E^3_T.

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